Boundary singularities of positive solutions of quasilinear Hamilton-Jacobi equations

Marie-Françoise Bidaut-Véron, Marta Garcia-Huidobro, Laurent Véron

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Abstract We study the boundary behaviour of the solutions of (E) $-\Delta_p u + |\nabla u|^q = 0$ in a domain $\Omega \subset \mathbb{R}^N$, when $N \geq p > q > p-1$. We show the existence of a critical exponent $q_* < p$ such that if $p-1 < q < q_*$ there exist positive solutions of (E) with an isolated singularity on $\partial\Omega$ and that these solutions belong to two different classes of singular solutions. If $q_* \leq q < p$ no such solution exists and actually any boundary isolated singularity of a positive solution of (E) is removable. We prove that all the singular positive solutions are classified according the two types of singular solutions that we have constructed.

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Key words. p-Laplace operator; singularities; spherical p-harmonic equations; Leray-Lions operators; Schauder fixed point theorem.

Contents

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1 Introduction

Let $N \geq p > 1$, $q > p-1$ and $\Omega \subset \mathbb{R}^N$ $(N > 1)$ be a C^2 bounded domain such that $0 \in \partial\Omega$. In this article we study the boundary behavior at 0 of nonnegative functions $u \in C^1(\Omega) \cap C(\overline{\Omega} \setminus \{0\})$ which satisfy

$$-\Delta_p u + |\nabla u|^q = 0 \qquad \text{in } \Omega, \tag{1.1}$$

where $\Delta_p u := \operatorname{div}\left(|\nabla u|^{p-2}\nabla u\right)$. The two main questions we consider are as follows:

Q-1- Existence of positive solutions of (1.1).

Q-2- Description of positive solutions with an isolated boundary singularity at 0.

When $p = 2$ a fairly complete description of positive solutions of

$$-\Delta u + |\nabla u|^q = 0 \tag{1.2}$$

in Ω is provided by Nguyen-Phuoc and Véron [11]. In particular they prove the following series of results in the range of values $1 < q < 2$.

1- Any signed solution of (1.3) verifies the estimates

$$|\nabla u(x)| \leq c_{N,q}\,(d(x))^{-\frac{1}{q-1}} \qquad \forall x \in \Omega, \tag{1.3}$$

where $d(x) = \operatorname{dist}(x, \partial\Omega)$. As a consequence, if $u \in C(\overline{\Omega} \setminus \{0\})$ is a solution which vanishes on $\partial\Omega \setminus \{0\}$, it satisfies

$$|u(x)| \leq c_{q,\Omega} d(x)|x|^{-\frac{1}{q-1}} \qquad \forall x \in \Omega. \tag{1.4}$$

2- If $\frac{N+1}{N} \leq q < 2$ any positive solution of (1.3) in Ω which vanishes on $\partial\Omega \setminus \{0\}$ is identically 0. *An isolated boundary point is a removable singularity for* (1.2).

3- If $1 < q < \frac{N+1}{N}$ and $k > 0$ there exists a unique positive solution $u := u_k$ of (1.2) in Ω which vanishes on $\partial\Omega \setminus \{0\}$ and satisfies $u(x) \sim c_N k P^\Omega(x, 0)$ as $x \to 0$, where P^Ω is the Poisson kernel in $\Omega \times \partial\Omega$.

4- If $1 < q < \frac{N+1}{N}$ there exists a unique positive solution u of (1.2) in the half-space $\mathbb{R}^N_+ := \{x = (x', x_N) : x' \in \mathbb{R}^{N-1}, x_N > 0\}$ under the form $u(x) = |x|^{-\frac{2-q}{q-1}}\omega(|x|^{-1}x)$ which vanishes on $\partial\mathbb{R}^N_+ \setminus \{0\}$. The function ω is the unique positive solution of

$$\begin{aligned} -\Delta'\omega + \left((\tfrac{2-q}{q-1})^2\omega^2 + |\nabla'\omega|^2\right)^{\frac{q}{2}} - \lambda_{N,q}\omega = 0 \qquad &\text{in } S^{N-1}_+, \\ \omega = 0 \qquad &\text{in } \partial S^{N-1}_+, \end{aligned} \tag{1.5}$$

where S^{N-1} is the unit sphere of $\mathbb{R}^N$, $\partial S^{N-1}_+ = \partial\mathbb{R}^N_+ \cap S^{N-1}$, Δ' the Laplace-Beltrami operator and $\lambda_{N,q} > 0$ an explicit constant.

5- If $1 < q < \frac{N+1}{N}$ and u is a positive solution of (1.3) in Ω, which is continuous in $\overline{\Omega} \setminus \{0\}$ and vanishes on $\partial\Omega \setminus \{0\}$ the following dichotomy occurs:

(i) either $u(x) \sim |x|^{-\frac{2-q}{q-1}}\omega(|x|^{-1}x)$ as $x \to 0$,

(ii) or $u(x) \sim kc_N P^\Omega(x, 0)$ as $x \to 0$ for some $k \geq 0$.

The aim of this article is to extend to the quasilinear case $1 < p \leq N$ the above mentioned results. The following pointwise gradient estimate valid for any signed solution u of (1.1) has been proved in [3]: if $0 < p-1 < q$ there exists a constant $c_{N,p,q} > 0$ such that

$$|\nabla u(x)| \leq c_{N,p,q}(d(x))^{-\frac{1}{q+1-p}} \qquad \forall x \in \Omega. \tag{1.6}$$

As a consequence, any solution $u \in C^1(\overline{\Omega} \setminus \{0\}$ satisfies

$$|u(x)| \leq c_{p,q,\Omega} d(x)\,|x|^{-\frac{1}{q+1-p}} \qquad \forall x \in \Omega. \tag{1.7}$$

Concerning boundary singularities, the situation is much more complicated than in the case $p = 2$ and the threshold of critical exponent less explicit. We first consider the problem in $\mathbb{R}^N_+$. Assuming $p-1 < q \leq p$, separable solutions of (1.1) in $\mathbb{R}^N_+$ vanishing on $\partial\mathbb{R}^N_+ \setminus \{0\}$ can be looked for in spherical coordinates $(r,\sigma) \in \mathbb{R}^*_+ \times S^{N-1}$ (we denote $\mathbb{R}^*_+ = (0,\infty)$) under the form

$$u(x) = u(r,\sigma) = r^{-\beta_q}\omega(\sigma), \quad r > 0\,,\ \sigma \in S^{N-1}_+ := \{S^{N-1} \cap \mathbb{R}^N_+\}. \tag{1.8}$$

Then ω is solution of the following problem

$$\begin{aligned} -div'\left(\left(\beta_q^2\omega^2 + |\nabla'\omega|^2\right)^{\frac{p-2}{2}}\nabla'\omega\right) - \beta_q\Lambda_{\beta_q}\left(\beta_q^2\omega^2 + |\nabla'\omega|^2\right)^{\frac{p-2}{2}}\omega & \\ + \left(\beta_q^2\omega^2 + |\nabla'\omega|^2\right)^{\frac{q}{2}} = 0 \quad \text{in } S^{N-1}_+ & \\ \omega = 0 \quad \text{on } \partial S^{N-1}_+, & \end{aligned} \tag{1.9}$$

where

$$\beta_q = \frac{p-q}{q+1-p} \text{ and } \Lambda_{\beta_q} = \beta_q(p-1) + p - N, \tag{1.10}$$

and ∇' is the covariant derivative on S^{N-1} identified to the tangential gradient thanks to the canonical isometrical imbedding of S^{N-1} into $\mathbb{R}^N$, and div' the divergence operator acting on vector fields on S^{N-1}.

The existence of a positive solution to this problem cannot be separated from the problem of existence of *separable p-harmonic functions* which are p-harmonic in $\mathbb{R}^N_+$ which vanish on $\partial\mathbb{R}^N_+ \setminus \{0\}$ and have the form $\Psi(x) = \Psi(r,\sigma) = r^{-\beta}\psi(\sigma)$ for some real number β. Necessarily such a ψ must satisfy

$$\begin{aligned} -div'\left(\left(\beta^2\psi^2 + |\nabla'\psi|^2\right)^{\frac{p-2}{2}}\nabla'\psi\right) - \beta\Lambda_\beta\left(\beta^2\psi^2 + |\nabla'\psi|^2\right)^{\frac{p-2}{2}}\psi = 0 \qquad & \text{in } S^{N-1}_+ \\ \psi = 0 \qquad & \text{on } \partial S^{N-1}_+, \end{aligned} \tag{1.11}$$

where $\Lambda_\beta = \beta(p-1) + p - N$. We will refer to (1.11) as *the spherical p-harmonic eigenvalue problem*. The study of this problem has been initiated in the 2-dim case by Krol [8] ($\beta < 0$) and Kichenassamy and Véron [9] ($\beta > 0$). In this case ω satisfies a completely integrable second order differential equation. In the case where S^{N-1}_+ is replaced by a smooth domain $S \subset S^{N-1}$ with $N \geq 3$, Tolksdorf [14] proved the existence of a unique couple $(\tilde{\beta}_s, \tilde{\psi}_s)$ where $\tilde{\beta}_s < 0$ and $\tilde{\psi}_s$ has constant sign and is defined up to an homothety. Recently Porretta and Véron [12] gave a simpler and more general proof of the existence of two couples $(\tilde{\beta}_s, \tilde{\psi}_s)$ and $(\beta_{*\,s}, \psi_{*\,s})$ where $\beta_{*\,s} > 0$ and $\tilde{\psi}_s$ and $\psi_{*\,s}$ are positive solutions of (1.11) with

$\beta = \tilde{\beta}_s$ and $\beta = \beta_{*\,s}$ respectively and are unique up to a multiplication by a real number. When $p = 2$ this problem is an eigenvalue problem for the Laplace-Beltrami operator on a subdomain of S^{N-1}. If $S = S^{N-1}_+$, $\tilde{\beta}_s$ and $\beta_{*\,s}$ are respectively denoted by $\tilde{\beta}$ and β_* and accordingly $\tilde{\psi}_s$ and $\psi_{*\,s}$ by $\tilde{\psi}$ and ψ_*. Since $x \mapsto x_N$ is p-harmonic, $\tilde{\beta} = -1$. Except in the cases $N = 2$ where it is the positive root of some algebraic equation of degree 2, $p = 2$ where it is $N - 1$ and $p = N$ where it is 1, the value of β_* is unknown besides the straightforward estimate $\beta_* \geq \max\{1, \frac{N-p}{p-1}\}$. Using the fact that ψ_* depends only on the azimuthal variable and satisfies a differential equation, we prove in Appendix II the following new estimate:

Theorem A *Let* $1 < p \leq N$.
(i) If $2 \leq p \leq N$*, then* $\beta_* \leq \frac{N-1}{p-1}$ *with equality only if* $p = 2$ *or* N.
(ii) If $1 \leq p < 2$*, then* $\beta_* > \frac{N-1}{p-1}$.

The p-harmonic function $\Psi_*(x) = \Psi_*(r, \sigma) = r^{-\beta_*}\psi_*(\sigma)$ endows the role of a Poisson kernel. To this exponent β_* is associated the critical value q_* of q defined by $\beta_* = \beta_q$, or equivalently

$$q_* := \frac{\beta_*(p-1) + p}{\beta_* + 1} = p - \frac{\beta_*}{\beta_* + 1}. \tag{1.12}$$

The following result characterizes strong singularities.

Theorem B *Let* $0 < p - 1 \leq N$*, then*
(i) If $p - 1 < q < q_*$ *problem* (1.9) *admits a unique positive solution* ω_*.
(ii) If $q_* \leq q < p$ *problem* (1.9) *admits no positive solution.*

This critical exponent corresponds to the threshold of criticality for boundary isolated singularities.

Theorem C *Assume* $q_* \leq q < p \leq N$*. If* $u \in C^1(\overline{\Omega} \setminus \{0\})$ *is a nonnegative solution of* (1.1) *in* Ω *which vanishes on* $\partial\Omega \setminus \{0\}$*, it is identical zero.*

As in the case $p = 2$, there exist positive solutions (1.1) in Ω with weak boundary singularities which are characterized by their blow-up near the singularity. By opposition to the case $p = 2$ where existence is obtained by use of a weak formulation of the boundary value problem, combined with uniform integrability of the absorption term thanks to Poisson kernel estimates (see [11]), this approach cannot be performed in the case $p \neq 2$; the obtention of solutions with weak singularities necessitates a very long and delicate construction of subsolutions and supersolutions. Furthermore, when $p \neq N$, the construction is done only if Ω is locally an hyperplane near 0.

In the sequel we denote by $B_R(a)$ the open ball of center a and radius $R > 0$ and $B_R = B_R(0)$. We also set $B^+_R(a) := \mathbb{R}^N_+ \cap B_R(a)$, $B^+_R := \mathbb{R}^N_+ \cap B_R$, $B^-_R(a) := \mathbb{R}^N_- \cap B_R(a)$ and $B^-_R := \mathbb{R}^N_- \cap B_R$, where $\mathbb{R}^N_- := \{x = (x', x_N) : x' \in \mathbb{R}^{N-1}, x_N < 0\}$. If Ω is an open domain and $R > 0$, we put $\Omega_R = \Omega \cap B_R$.

Theorem D *Let* $\Omega \subset \mathbb{R}^N_+$ *be a bounded domain such that* $0 \in \partial\Omega$*. Assume there exists* $\delta > 0$ *such that* $\Omega_\delta = B^+_\delta$ *and* $0 < p - 1 < q < q_* < p \leq N$*. Then for any* $k > 0$ *there exists a unique* $u := u_k \in C^1(\overline{\Omega} \setminus \{0\})$*, solution of* (1.1) *in* Ω*, vanishing on* $\partial\Omega \setminus \{0\}$ *and such that*

$$\lim_{\substack{x \to 0 \\ \frac{x}{|x|} \to \sigma \in S^{N-1}_+}} |x|^{\beta_*} u_k(x) = k\psi_*(\sigma). \tag{1.13}$$

Furthermore $\lim_{k\to\infty} u_k = u_\infty$ *and*

$$\lim_{\substack{x \to 0 \\ \frac{x}{|x|} \to \sigma \in S_+^{N-1}}} |x|^{\beta_q} u_\infty(x) = \psi_*(\sigma). \tag{1.14}$$

When $p = N$, then $q_* = N - \frac{1}{2}$; in such a range of values we use the conformal invariance of Δ_N and prove that the previous result holds if Ω is *any* C^2 domain. Finally, the isolated singularities of positive solutions of (1.1) are completely described by the two types of singular solutions obtained in the previous theorem and we prove:

Theorem E *Let* Ω *be a bounded domain such that* $0 \in \partial\Omega$. *Assume there exists* $\delta > 0$ *such that* $\Omega_\delta = B_\delta^+$ *and* $0 < p - 1 < q < q_* < p \leq N$. *If* $u \in C^1(\overline{\Omega} \setminus \{0\})$ *is a positive solution of* (1.1) *in* Ω *which vanishes on* $\partial\Omega \setminus \{0\}$, *then*

(i) either there exists $k \geq 0$ *such that*

$$\lim_{\substack{x \to 0 \\ \frac{x}{|x|} \to \sigma \in S_+^{N-1}}} |x|^{\beta_*} u(x) = k\psi_*(\sigma); \tag{1.15}$$

(ii) or

$$\lim_{\substack{x \to 0 \\ \frac{x}{|x|} \to \sigma \in S_+^{N-1}}} |x|^{\beta_q} u(x) = \psi_*(\sigma). \tag{1.16}$$

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2 A priori estimates

2.1 The gradient estimates and its applications

We recall the following estimate and its consequences which are proved in [3].

Proposition 2.1. *Assume* $q > p - 1$ *and* u *is a* C^1 *solution of* (1.1) *in a domain* Ω. *Then*

$$|\nabla u(x)| \leq c_{N,p,q}(d(x))^{-\frac{1}{q+1-p}} \qquad \forall x \in \Omega. \tag{2.1}$$

The first application is a pointwise upper bound for solutions with isolated singularities.

Corollary 2.2. *Assume* $q > p - 1 > 0$, $R^* > 0$ *and* Ω *is a domain containing* 0 *such that* $d(0) \geq 2R^*$. *Then for any* $x \in B_{R^*} \setminus \{0\}$, *and* $0 < R \leq R^*$, *any* $u \in C^1(\Omega \setminus \{0\})$ *solution of* (1.1) *in* $\Omega \setminus \{0\}$ *satisfies*

$$|u(x)| \leq c_{N,p,q} \left| |x|^{\frac{q-p}{q+1-p}} - R^{\frac{q-p}{q+1-p}} \right| + \max\{|u(z)| : |z| = R\}, \tag{2.2}$$

if $p \neq q$, *and*

$$|u(x)| \leq c_{N,p} \left(\ln R - \ln |x|\right) + \max\{|u(z)| : |z| = R\}, \tag{2.3}$$

if $p = q$.

The second application corresponds to solutions with boundary blow-up. For $\delta > 0$ small enough we set $\Omega_\delta := \{z \in \Omega : d(z) < \delta\}$.

Corollary 2.3. *Assume $q > p-1 > 0$, Ω is a bounded domain with a C^2 boundary. Then there exists $\delta_1 > 0$ which depends only on Ω such that any $u \in C^1(\Omega)$ solution of* (1.1) *in Ω satisfies*

$$|u(x)| \leq c_{N,p,q} \left| (d(x))^{\frac{q-p}{q+1-p}} - \delta_1^{\frac{q-p}{q+1-p}} \right| + \max\{|u(z)| : d(z) = \delta_1\} \quad \forall x \in \Omega_{\delta_1} \tag{2.4}$$

if $p \neq q$, and

$$|u(x)| \leq c_{N,p,q} \left(\ln \delta_1 - \ln d(x)\right) + \max\{|u(z)| : d(z) = \delta_1\} \quad \forall x \in \Omega_{\delta_1} \tag{2.5}$$

if $p = q$.

Remark. As a consequence of (2.4) there holds for $p > q > p-1$

$$u(x) \leq \left(c_{N,p,q} + K \max\{|u(z)| : d(z) \geq \delta_1\}\right) (d(x))^{\frac{q-p}{q+1-p}} \quad \forall x \in \Omega \tag{2.6}$$

where $K = (\mathrm{diam}(\Omega))^{\frac{p-q}{q+1-p}}$, with the standard modification if $p = q$.

As a variant of Corollary 2.3 the following upper estimate of solutions in an exterior domain will be used in the sequel.

Corollary 2.4. *Assume $q > p-1 > 0$, $R > 0$ and $u \in C^1(B^c_{R_0})$ is any solution of* (1.1) *in $B^c_{R_0}$. Then for any $R > R_0$ there holds*

$$|u(x)| \leq c_{N,p,q} \left| (|x| - R_0)^{\frac{q-p}{q+1-p}} - (R - R_0)^{\frac{q-p}{q+1-p}} \right| + \max\{|u(z)| : |z| = R\} \quad \forall x \in B^c_R \tag{2.7}$$

if $p \neq q$ and

$$|u(x)| \leq c_{N,p,q} \left(\ln(|x| - R_0) - \ln(R - R_0)\right) + \max\{|u(z)| : |z| = R\} \quad \forall x \in B^c_R \tag{2.8}$$

if $p = q$.

Proof. The proof is a consequence of the identity

$$u(x) = u(z) + \int_0^1 \frac{d}{dt} u(tx + (1-t)z) dt = \int_0^1 \langle \nabla u(tx + (1-t)z), x - z \rangle dt$$

where $z = \frac{R}{|x|} x$. Since by (2.1)

$$|\nabla u(tx + (1-t)z)| \leq C_{N,p,q} (t\,|x| + (1-t)R - R_0)^{-\frac{1}{q+1-p}},$$

(2.7) and $(2.8$ $)$ follow by integration. $\square$

2.2 Boundary a priori estimates

The next result is the extension to isolated boundary singularities of a previous regularity estimate dealing with singularity in a domain proved in [3, Lemma 3.10].

Lemma 2.5. *Assume $p-1<q<p$, Ω is a bounded C^2 domain such that $0\in\partial\Omega$. Let $u\in C^1(\overline{\Omega}\setminus\{0\})$ be a solution of (1.1) in Ω which vanishes on $\partial\Omega\setminus\{0\}$ and satisfies*

$$|u(x)|\leq\phi(|x|)\qquad\forall x\in\Omega,\tag{2.9}$$

where $\phi:\mathbb{R}_+^*\mapsto\mathbb{R}_+$ is continuous, nonincreasing and satisfies

$$\phi(rs)\leq\gamma\phi(r)\phi(s)\quad\textit{and}\quad r^{\frac{p-q}{q+1-p}}\phi(r)\leq c,\tag{2.10}$$

for some $\gamma,c>0$ and any $r,s>0$. There exist $\alpha\in(0,1)$ and $c_1=c_1(p,q,\Omega)>0$ such that

$$\begin{array}{lll}(i) & |\nabla u(x)|\leq c_1\phi(|x|)\,|x|^{-1} & \forall x\in\Omega,\\ (ii) & |\nabla u(x)-\nabla u(y)|\leq c_1\phi(|x|)\,|x|^{-1-\alpha}\,|x-y|^{\alpha} & \forall x,y\in\Omega,\ |x|\leq|y|\,.\end{array}\tag{2.11}$$

Furthermore

$$u(x)\leq c_1\phi(|x|)\frac{d(x)}{|x|}\qquad\forall x\in\Omega.\tag{2.12}$$

Proof. For $\ell>0$, we set $\Omega^\ell:=\frac{1}{\ell}\Omega$. If $\ell\in(0,1]$ the curvature of $\partial\Omega^\ell$ remains uniformly bounded. As in [5, p 622], there exists $0<\delta_0\leq1$ and an involutive diffeomorphism ψ from $\overline{B}_{\delta_0}\cap\overline{\Omega}^{\delta_0}$ into $\overline{B}_{\delta_0}\cap(\Omega^{\delta_0})^c$ which is the identity on $\overline{B}_{\delta_0}\cap\partial\Omega^{\delta_0}$ and such that $D\psi(\xi)$ is the symmetry with respect to the tangent plane $T_\xi\partial\Omega$ for any $\xi\in\partial\Omega\cap\overline{B}_{\delta_0}$. We extend any function v defined in $\overline{B}_{\delta_0}\cap\overline{\Omega}^{\delta_0}$ and vanishing on $\overline{B}_{\delta_0}\cap\partial\Omega^{\delta_0}$ into a function $\tilde v$ defined in $\overline{B}_{\delta_0}$ by

$$\tilde v(x)=\left\{\begin{array}{ll}v(x) & \text{if } x\in\overline{B}_{\delta_0}\cap\overline{\Omega}^{\delta_0}\\ -v\circ\psi(x) & \text{if } x\in\overline{B}_{\delta_0}\cap(\Omega^{\delta_0})^c,\end{array}\right.\tag{2.13}$$

If $v\in C^1(\overline{B}_{\delta_0}\cap\overline{\Omega}^{\delta_0})$ is a solution of (1.1) in $B_{\delta_0}\cap\Omega^{\delta_0}$ which vanishes on $\partial\Omega^{\delta_0}\cap\overline{B}_{\delta_0}$, $\tilde v$ satisfies

$$-\sum_j\frac{\partial}{\partial x_j}\tilde A_j(x,\nabla\tilde v)+B(x,\nabla\tilde v)=0\quad\text{in } B_{\delta_0}.\tag{2.14}$$

As in [5, (2.37)] the A_j and B satisfy the following estimates

$$\begin{array}{ll}(i) & \tilde A_j(x,0)=0\\ (ii) & \displaystyle\sum_{i,j}\frac{\partial}{\partial\eta_i}\tilde A_j(x,\eta)\xi_i\xi_j\geq C_1\,|\eta|^{p-1}\,|\xi|^2\\ (iii) & \displaystyle\sum_{i,j}\left|\frac{\partial}{\partial\eta_j}\tilde A_j(x,\eta)\right|\leq C_2\,|\eta|^{p-2}\,,\end{array}\tag{2.15}$$

and

$$|B(x,\eta)| \leq C_3(1+|\eta|)^p, \tag{2.16}$$

where the C_j are positive constants. These estimates are the ones needed to apply Tolksdorf's result [15, Th 1,2]. There exists a constant C, such that for any ball $\overline{B}_{3R} \subset \overline{B}_{\delta_0}$, there holds

$$\|\nabla \tilde{v}\|_{L^\infty(B_R)} \leq C, \tag{2.17}$$

where C depends on the constants C_k ($k = 1,2,3$), N, p and $\|\tilde{v}\|_{L^\infty(B_{3R})}$. We define

$$\Phi_\ell[u](y) := u_\ell = \frac{1}{\phi(\ell)} u(\ell y) \qquad \forall y \in \Omega^\ell. \tag{2.18}$$

Then

$$|u_\ell(y)| \leq \frac{\phi(\ell\,|y|)}{\phi(\ell)} \leq \gamma\phi(|y|) \qquad \forall y \in \Omega^\ell \tag{2.19}$$

and

$$-\Delta_p u_\ell + (\ell^{\beta_q}\phi(\ell))^{q+1-p}\,|\nabla u_\ell|^q = 0 \qquad \text{in } \Omega^\ell. \tag{2.20}$$

Using formula (2.13) we extend u_ℓ into a function $\tilde{u}_\ell$ which satisfies

$$-\sum_j \frac{\partial}{\partial y_j}\tilde{A}_j(y,\nabla\tilde{u}_\ell) + (\ell^{\beta_q}\phi(\ell))^{q+1-p} B(y,\nabla\tilde{u}_\ell) = 0 \quad \text{in } B_{\delta_0}. \tag{2.21}$$

For $0 < |x| < \delta_0$ there exists $\ell \in (0,2)$ such that $\frac{\delta_0\ell}{2} \leq |x| \leq \delta_0\ell$. Then $y \mapsto \tilde{u}_\ell(y)$ with $y = \frac{x}{\ell}$ satisfies (2.21) in B_{δ_0} and $|\tilde{u}_\ell(y)| \leq \gamma_*\phi(|y|)$ since ψ is a diffeomorphism and $D\psi(\xi) \in O(N)$ for any $\xi \in \partial\Omega \cap B_{\delta_0}$. The function $\tilde{u}_\ell$ remains bounded on any ball $B_{3R}(z) \subset \Gamma := \{y \in \mathbb{R}^N : \frac{\delta_0}{2} < |y| < \delta_0\}$, therefore $|\nabla\tilde{u}_\ell(y)| \leq c$ for any $y \in B_R(z)$, for some constant $c > 0$. This implies

$$|\nabla u(x)| \leq c\gamma_*\delta_0\phi(\tfrac{2}{\delta_0})\phi(|x|)|x|^{-1} \quad \forall x \in \Omega \cap B_{\delta_0}, \tag{2.22}$$

which is (2.11)-(i). Moreover, by standard regularity estimates [10], there exists $\alpha \in (0,1)$ such that $|\nabla\tilde{u}_\ell(y) - \nabla\tilde{u}_\ell(y')| \leq c\,|y-y'|^\alpha$ for all y and y' belonging to $B_R(z)$. This implies (2.11)-(ii).

Next we prove (2.12). Let $0 < \delta_1 \leq \delta_0$ such that at any boundary point z there exist two closed balls of radius δ_1 tangent to $\partial\Omega$ at z and which are included in $\Omega \cup \{z\}$ and in $\overline{\Omega}^c \cup \{z\}$ respectively (δ_1 corresponds to the maximal radius of the interior and exterior sphere condition). Let $x \in \Omega$ such that $d(x) \leq \delta_1$ (this is not a loss of generality) and z_x be the projection of x on $\partial\Omega$. We first assume that x does not belong to the cone $\Sigma_{\frac{\pi}{4}}$ with vertex 0, axis $-\mathbf{n}_0$, where $\mathbf{n}_0$ is the normal outward unit vector at 0, and angle $\frac{\pi}{4}$. Consider the path ζ from z_x to x defined by $\zeta(t) = tx + (1-t)z_x$ with $0 \leq t \leq 1$. Then

$$u(x) = \int_0^1 \frac{d}{dt} u \circ \zeta(t)dt = \int_0^1 \langle \nabla u \circ \zeta(t), x - z_x \rangle dt \tag{2.23}$$

Thus, by the Cauchy-Schwarz inequality, using (2.11),

$$|u(x)| \leq c_1 d(x) \int_0^1 \frac{\phi(|\zeta(t)|)}{|\zeta(t)|} dt. \tag{2.24}$$

Since $x \notin \Sigma_{\frac{\pi}{4}}$, $\zeta(t) \notin \Sigma_{\frac{\pi}{4}}$ and there exists $c_2 > 0$ depending on Ω such that $c_2^{-1}|x| \leq |\zeta(t)| \leq c_2|x|$ for all $0 \leq t \leq 1$. Therefore $\phi(|\zeta(t)|) \leq \phi(c_2|x|) \leq \gamma\phi(c_2)\phi(|x|)$ by (2.10). This implies

$$|u(x)| \leq \gamma c_1 c_2 \phi(c_2)\frac{d(x)\phi(|x|)}{|x|} \tag{2.25}$$

by (2.12) whenever $x \notin \Sigma_{\frac{\pi}{4}}$. When $x \in \Sigma_{\frac{\pi}{4}}$ then $d(x) \leq |x| \leq c_3 d(x)$ where $c_3 > 0$ depends on the curvature of $\partial\Omega$. Then (2.9) combined with (2.10) implies the claim. □

Lemma 2.6. *Assume $p-1 < q \leq p$, Ω is a bounded C^2 domain such that $0 \in \partial\Omega$ and $R_0 = \max\{|z| : z \in \Omega\}$. If $u \in C(\overline{\Omega}\setminus\{0\}) \cap C^1(\Omega)$ is a positive solution of (1.1) which vanishes on $\partial\Omega\setminus\{0\}$, it satisfies*

$$u(x) \leq \begin{cases} c_2\left(|x|^{\frac{q-p}{q+1-p}} - R_0^{\frac{q-p}{q+1-p}}\right) & \text{if } q < p \\ (p-1)\ln\left(\frac{R_0}{|x|}\right) & \text{if } q = p \end{cases} \tag{2.26}$$

for all $x \in \Omega$, where $c_2 = c_2(p,q) > 0$.

Proof. For $\epsilon > 0$ we denote by $P_\epsilon : \mathbb{R} \mapsto \mathbb{R}_+$ the function defined by

$$P_\epsilon(r) = \begin{cases} 0 & \text{if } 0 \leq r \leq \epsilon \\ -\frac{r^4}{2\epsilon^3} + \frac{3r^3}{\epsilon^2} - \frac{6r^2}{\epsilon} + 5r - \frac{3\epsilon}{2} & \text{if } \epsilon < r < 2\epsilon \\ r - \frac{3\epsilon}{2} & \text{if } r \geq 2\epsilon, \end{cases} \tag{2.27}$$

and by u_ϵ the extension of $P_\epsilon(u)$ by zero outside Ω. There exists R_0 such that $\Omega \subset B_{R_0}$. Since $0 \leq P_\epsilon(r) \leq |r|$ and P_ϵ is convex, $u_\epsilon \in C(\mathbb{R}^N\setminus\{0\}) \cap W^{1,p}_{loc}(\mathbb{R}^N\setminus\{0\})$ and

$$-\Delta_p u_\epsilon + |\nabla u_\epsilon|^q \leq 0 \qquad \text{in } \mathbb{R}^N.$$

Let $R > R_0$. If $p-1 < q < p$

$$U_{\epsilon,R}(|x|) = c_2\left((|x|-\epsilon)^{\frac{q-p}{q+1-p}} - (R-\epsilon)^{\frac{q-p}{q+1-p}}\right) \qquad \text{in } B_R \setminus B_\epsilon, \tag{2.28}$$

with $c_2 = (p-q)^{-1}(q+p-1)^{\frac{q-p}{q+1-p}}$. Then $-\Delta_p U_{\epsilon,R} + |\nabla U_{\epsilon,R}|^q \geq 0$. Since u_ϵ vanishes on ∂B_R and is finite on ∂B_ϵ, it follows $u_\epsilon \leq U_{\epsilon,R}$. Letting successively $\epsilon \to 0$ and $R \to R_0$ yields to (2.26). If $q = p$ we take

$$U_{\epsilon,R}(|x|) = (p-1)\ln\left(\frac{R-\epsilon}{|x|-\epsilon}\right) \qquad \text{in } B_R \setminus B_\epsilon, \tag{2.29}$$

which turns out to be a supersolution of (1.1); the end of the proof is similar. □

As a consequence of Lemma 2.5 and Lemma 2.6, we obtain.

Corollary 2.7. *Let p, q Ω and u be as in Lemma 2.6. Then there exists a constant $c_3 = c_3(p,q,\Omega) > 0$ such that*

$$|\nabla u(x)| \leq c_3|x|^{-\frac{1}{q+1-p}} \qquad \forall x \in \Omega \tag{2.30}$$

and

$$u(x) \leq c_3 d(x)|x|^{-\frac{1}{q+1-p}} \qquad \forall x \in \overline{\Omega}\setminus\{0\}. \tag{2.31}$$

Remark. If Ω is locally flat near 0, then estimates (2.30) and (2.31) are valid without any sign assumption on u. More precisely, if $\partial\Omega\cap B_{\delta_0}=T_0\partial\Omega\cap B_{\delta_0}$ we can perform the reflection of u through the tangent plane $T_0\partial\Omega$ to $\partial\Omega$ at 0 and the new function $\tilde u$ is a solution of (1.1) in $B_{\delta_0}\setminus\{0\}$. By Proposition 2.1, it satisfies

$$|\nabla\tilde u(x)|\leq c_{N,p,q}\,|x|^{-\frac{1}{q+1-p}}\qquad\forall x\in B_{\frac{\delta_0}{2}}\setminus\{0\}. \tag{2.32}$$

Integrating this relation as in [3], we derive that for any $x\in B_{\frac{\delta_0}{2}}\cap\Omega$, there holds

$$|u(x)|\leq\left\{\begin{array}{ll} c_{N,p,q}\left(|x|^{-\beta_q}-(\frac{\delta_0}{2})^{-\beta_q}\right)+\max\{|u(z)|:|z|=\frac{\delta_0}{2}\} & \text{if } p\neq q\\ c_{N,p}\ln\left(\frac{\delta_0}{2|x|}\right)+\max\{|u(z)|:|z|=\frac{\delta_0}{2}\} & \text{if } p=q. \end{array}\right. \tag{2.33}$$

In the next result we allow the boundary singular set to be a compact set.

Proposition 2.8. *Let $p-1<q<p$ and δ_1 as above. There exist $r^*\in(0,\delta_1]$ and $c_4=c_4(N,p,q)>0$ such that for any nonempty compact set $K\subset\partial\Omega$, $K\neq\partial\Omega$ and any positive solution $u\in C(\overline{\Omega}\setminus K)\cap C^1(\Omega)$ of (1.1) which vanishes on $\partial\Omega\setminus K$, there holds*

$$u(x)\leq c_4 d(x)(d_K(x))^{-\frac{1}{q+1-p}}\qquad\forall x\in\partial\Omega \text{ s.t. } d(x)\leq r^*, \tag{2.34}$$

where $d_K(x)=\operatorname{dist}(x,K)$.

Proof. Step 1: Tangential estimates. Let $x\in\Omega$ such that $d(x)\leq\delta_1$. We denote by $\sigma(x)$ the projection of x onto $\partial\Omega$, unique since $d(x)\leq\delta_1$. Let $r\,,r',\tau>0$ such that $\frac{3}{4}r<r'<\frac{7}{8}r$ and $0<\tau\leq\frac{r'}{2}$ and put $\omega_{\tau,x}=\sigma(x)+\tau\mathbf{n}_{\sigma(x)}$. Since $\partial\Omega$ is C^2, there exists $0<r^*\leq\delta_1$ depending on Ω such that $d_K(\omega_{\tau,x})>\frac{7}{8}r$ whenever $d(x)\leq r^*$. Let $a>0$ and $b>0$ to be specified later on; we define $\tilde v(s)=a(r'-s)^{\frac{q-p}{q+1-p}}-b$ and $v(y)=\tilde v(|y-\omega_{\tau,x}|)$ in $[0,r')$ and $B_{r'}(\omega_{\tau,x})$ respectively. Then

$$|\tilde v'|^{p-2}\left(|\tilde v'|^{q+2-p}-(p-1)\tilde v''-\frac{N-1}{s}\tilde v'\right)=a^{p-1}\left(\frac{p-q}{q+1-p}\right)^{p-1}(r'-s)^{-\frac{q}{q+1-p}}X(s)$$

where

$$X(s)=\left(a\frac{p-q}{q+1-p}\right)^{q+1-p}-\frac{p-1}{q+1-p}-\frac{(N-1)(r'-s)}{s}.$$

For any $\tau\in(0,r')$ there exists $a>0$ such that

$$\left(a\frac{p-q}{q+1-p}\right)^{q+1-p}\geq\frac{p-1}{q+1-p}+\frac{(N-1)(r'-s)}{s}\qquad\forall\tau\leq s\leq r'.$$

This implies

$$-\Delta_p v+|\nabla v|^q\geq 0\qquad\text{in } B_{r'}(\omega_{\tau,x})\setminus B_\tau(\omega_{\tau,x}). \tag{2.35}$$

Next we take $b=a(r'-\tau)^{\frac{q-p}{q+1-p}}$, thus $v=0$ on $\partial B_\tau(\omega_{\tau,x})$. Clearly $B_\tau(\omega_{\tau,x})\subset\overline{\Omega}^c$ since $\tau<\delta_1$. Therefore $v\geq 0=u$ on $\partial\Omega\cap B_{r'}(\omega_{\tau,x})$ and $u\leq v=\infty$ on $\Omega\cap\partial B_{r'}(\omega_{\tau,x})$. By the comparison principle, $v\geq u$ in $\Omega\cap B_{r'}(\omega_{\tau,x})$. In particular

$$u(x)\leq v(x)\leq a(r'-\tau-d(x))^{\frac{q-p}{q+1-p}}-a(r'-\tau)^{\frac{q-p}{q+1-p}}.$$

We take now $\tau = \frac{r'}{2}$ and $d(x) \leq \frac{r}{4}$ and we derive by the mean value theorem

$$u(x) \leq c_4' r'^{-\frac{1}{q+1-p}} d(x) = c_4' d(x)(d_K(x))^{-\frac{1}{q+1-p}}, \tag{2.36}$$

with $c_4' = c_4'(p,q) > 0$ Letting $r' \to \frac{7}{8}r$, we get (2.12).

Step 2: Global estimates. If $d(x) \geq \frac{1}{4}d_K(x)$, there holds

$$d(x)(d_K(x))^{-\frac{1}{q+1-p}} \geq 2^{-\frac{2}{q+1-p}}(d(x))^{\frac{q-p}{q+1-p}}.$$

Combining this inequality with (2.6) and obtain (2.34). □

Remark. Under the assumption of Proposition 2.8, it follows from the maximum principle that u is upper bounded in the set $\Omega'_{r^*} := \{x \in \Omega : d(x) > r^*\} = \Omega \setminus \overline{\Omega}_{r^*}$ by the solution w of

$$\begin{aligned} -\Delta_p w + |\nabla w|^q = 0 \qquad & \text{in } \Omega_{r^*} \\ w = c_4 d(x)(d_K(x))^{-\frac{1}{q+1-p}} \qquad & \text{in } \partial\Omega_{r^*}, \end{aligned} \tag{2.37}$$

and w itself is bounded by $d^* = \max\{cd(x)(d_K(x))^{-\frac{1}{q+1-p}} : d(x) = r^*\}$.

Next we prove a boundary Harnack inequality. We recall that δ_1 has been introduced at Corollary 2.3, and that the interior and exterior sphere conditions hold in the set $\{x \in \mathbb{R}^N : d(x) \leq \delta_1\}$.

Theorem 2.9. *Let $q > p-1$ and $0 \in \partial\Omega$. Then there exists $c_5 = c_5(N,p,q,\Omega) > 0$ such that for any positive solution $u \in C(\Omega \cup ((\partial\Omega \setminus \{0\}) \cap B_{2\delta_1}) \cap C^1(\Omega)$ of (1.1) in Ω, vanishing on $\partial\Omega \setminus \{0\}) \cap B_{2\delta_1}$, there holds*

$$\frac{u(y)}{c_5 d(y)} \leq \frac{u(x)}{d(x)} \leq c_5 \frac{u(y)}{d(y)} \tag{2.38}$$

for all $x, y \in B_{\frac{2\delta_1}{3}} \cap \Omega$ such that $\frac{1}{2}|x| \leq |y| \leq 2|x|$.

For proving Theorem 2.9 we need some intermediate lemmas. First we recall the following result from [1].

Lemma 2.10. *Assume that $a \in \partial\Omega$, $0 < r < \delta_1$ and $h > 1$ is an integer. There exists an integer N_0, depending only on δ_1, such that for any points x and y in $\Omega \cap B_{\frac{3r}{2}}(a)$ verifying $\min\{d(x), d(y)\} \geq r/2^h$, there exists a connected chain of balls $B_1, ..., B_j$ with $j \leq N_0 h$ such that*

$$\begin{aligned} & x \in B_1, y \in B_j, \quad B_i \cap B_{i+1} \neq \emptyset \textit{ for } 1 \leq i \leq j-1 \\ & \textit{and } 2B_i \subset B_{2r}(Q) \cap \Omega \textit{ for } 1 \leq i \leq j. \end{aligned} \tag{2.39}$$

The next result is a standard Harnack inequality.

Lemma 2.11. *Assume $a \in (\partial\Omega \setminus \{0\}) \cap B_{\frac{2\delta_1}{3}}$ and $0 < r \leq |a|/4$. Let $u \in C(\Omega \cup ((\partial\Omega \setminus \{0\}) \cap B_{2\delta_1})) \cap C^1(\Omega)$ be a positive solution of (1.1) vanishing on $(\partial\Omega \setminus \{0\}) \cap B_{2\delta_1}$. Then there exists a positive constant $c_6 > 1$ depending on N, p, q and δ_1 such that*

$$u(x) \leq c_6^h u(y), \tag{2.40}$$

for every $x, y \in B_{\frac{3r}{2}}(a) \cap \Omega$ such that $\min\{d(x), d(y)\} \geq r/2^h$ for some $h \in \mathbb{N}$.

Proof. For $\ell > 0$, we define $T_\ell[u]$ by

$$T_\ell[u](x) = \ell^{\frac{p-q}{q+1-p}} u(\ell x), \tag{2.41}$$

and we notice that if u satisfies (1.1) in Ω, then $T_\ell[u]$ satisfies the same equation in $\Omega^\ell := \ell^{-1}\Omega$. If we take in particular $\ell = |a|$, we can assume $|a| = 1$, thus the curvature of the domain $\Omega^{|a|}$ remains bounded. By Proposition 2.8

$$u(x) \leq c_6' \quad \forall x \in B_{2r}(a) \cap \Omega \tag{2.42}$$

where c_6' depends on N, q, δ_1. Then we proceed as in [11], using Lemma 2.10 and internal Harnack inequality as quoted in [16, Corollary 10]. □

Since the solutions are Hölder continuous, the following statement holds as in [16, Theorem 4.2]:

Lemma 2.12. *Let the assumptions on a and u of Lemma 2.11 be fulfilled. If $b \in \partial\Omega \cap B_r(a)$ and $0 < s \leq 2^{-1}r$, there exist two positive constants δ and c_7 depending on N, p, q and Ω such that*

$$u(x) \leq c_7 \frac{|x-b|^\delta}{s^\delta} \max\{u(z) : z \in B_r(b) \cap \Omega\} \tag{2.43}$$

for every $x \in B_s(b) \cap \Omega$.

As a consequence we derive the following Carleson type estimate.

Lemma 2.13. *Assume $a \in (\partial\Omega \setminus \{0\}) \cap B_{\frac{2\delta_1}{3}}$ and $0 < r \leq |a|/8$. Let $u \in C(\Omega \cup ((\partial\Omega \setminus \{0\}) \cap B_{2\delta_1})) \cap C^2(\Omega)$ be a positive solution of (1.1) vanishing on $(\partial\Omega \setminus \{0\}) \cap B_{2\delta_1}$. Then there exists a constant c_8 depending only on N, p and q such that*

$$u(x) \leq c_8 u(a - \tfrac{r}{2}\mathbf{n}_a) \quad \forall x \in B_r(a) \cap \Omega. \tag{2.44}$$

Proof. By Lemma 2.11 it is clear that for any integer h and $x \in B_r(a) \cap \Omega$ such that $d(x) \geq 2^{-h}r$, there holds

$$u(x) \leq c_6^h u(a - \tfrac{r}{2}\mathbf{n}_a). \tag{2.45}$$

Therefore u satisfies inequality (2.43) as any Hölder continuous function does. The proof that the constant is independent of r and u is more delicate. It is done in [1, Lemma 2.4] for linear equations, but it is based only on Lemma 2.12 and a geometric construction, thus it is also valid in our case. □

Lemma 2.14. *Assume $a \in (\partial\Omega \setminus \{0\}) \cap B_{\frac{2\delta_1}{3}}$ and $0 < r \leq |a|/8$. Let $u \in C(\Omega \cup ((\partial\Omega \setminus \{0\}) \cap B_{2\delta_1})) \cap C^2(\Omega)$ be a positive solution of (1.1) vanishing on $(\partial\Omega \setminus \{0\}) \cap B_{2\delta_1}$. Then there exist $\alpha \in (0, 1/2)$ and $c_9 > 0$ depending on N, p and q such that*

$$\frac{1}{c_9}\frac{t}{r} \leq \frac{u(b - t\mathbf{n}_b)}{u(a - \frac{r}{2}\mathbf{n}_a)} \leq c_9 \frac{t}{r} \tag{2.46}$$

for any $b \in B_r(a) \cap \partial\Omega$ and $0 \leq t < \frac{\alpha}{2}r$.

Proof. It is similar to the one of [11, Lemma 3.15]. □

Proof of Theorem 2.9. Assume $x \in B_{\frac{2\delta_1}{3}} \cap \Omega$ and set $r = \frac{|x|}{8}$.

Step 1: Tangential estimate: we suppose $d(x) < \frac{\alpha}{2} r$. Let $a \in \partial\Omega \backslash \{0\}$ such that $|a| = |x|$ and $x \in B_r(a)$. By Lemma 2.14,

$$\frac{8}{c_9} \frac{u(a - \frac{r}{2}\mathbf{n}_a)}{|x|} \leq \frac{u(x)}{d(x)} \leq 8c_9 \frac{u(a - \frac{r}{2}\mathbf{n}_a)}{|x|}. \tag{2.47}$$

We can connect $a - \frac{r}{2}\mathbf{n}_a$ with $-2r\mathbf{n}_0$ by m_1 (depending only on N) connected balls $B_i = B_{\frac{r}{4}}(x_i)$ with $x_i \in \Omega$ and $d(x_i) \geq \frac{r}{2}$ for every $1 \leq i \leq m_1$. It follows from (2.44) that

$$c_6^{-m_1} u(-2r\mathbf{n}_0) \leq u(a - \tfrac{r}{2}\mathbf{n}_a) \leq c_6^{m_1} u(-2r\mathbf{n}_0),$$

which, together with (2.47) leads to

$$\frac{1}{c_{10}} \frac{u(-2r\mathbf{n}_0)}{|x|} \leq \frac{u(x)}{d(x)} \leq c_{10} \frac{u(-2r\mathbf{n}_0)}{|x|}, \tag{2.48}$$

with $c_{10} = 8c_9 c_6^{m_1}$.

Step 2: Internal estimate: we suppose $d(x) \geq \frac{\alpha}{2} r$. We can connect $-2r\mathbf{n}_0$ with x by m_2 (depending only on N) connected balls $B'_i = B_{\frac{\alpha r}{4}}(x'_i)$ with $x'_i \in \Omega$ and $d(x'_i) \geq \frac{\alpha}{2} r$ for every $1 \leq i \leq m_2$. By Harnack and Carleson inequalities (2.40) and (2.44) and since $\frac{\alpha}{4}|x| < d(x) \leq |x|$, we get

$$\frac{\alpha}{4c_6'^{m_2}} \frac{u(-2r\mathbf{n}_0)}{|x|} \leq \frac{u(x)}{d(x)} \leq \frac{4c_6'^{m_2}}{\alpha} \frac{u(-2r\mathbf{n}_0)}{|x|}. \tag{2.49}$$

Step 3: End of proof. Suppose $\frac{|x|}{2} \leq s \leq 2|x|$, we can connect $-2r\mathbf{n}_Q$ with $-s\mathbf{n}_Q$ by m_3 (depending only on N) connected balls $B''_i = B_{\frac{r}{2}}(x''_i)$ with $x''_i \in \Omega$ and $d(x''_i) \geq r$ for every $1 \leq i \leq m_3$. This fact, jointly with (2.48) and (2.49), yields to

$$\frac{1}{c_{11}} \frac{u(-s\mathbf{n}_0)}{|x|} \leq \frac{u(x)}{d(x)} \leq c_{11} \frac{u(-s\mathbf{n}_0)}{|x|} \tag{2.50}$$

where $c_{11} = c_{11}(N, q, \Omega)$. Finally, if $y \in B_{\frac{2r_0}{3}} \cap \Omega$ satisfies $\frac{|x|}{2} \leq |y| \leq 2|x|$, then by applying twice (2.50) we get (2.38) with $c_5 = c_{11}^2$. □

The following inequality is a consequence of Theorem 2.9.

Corollary 2.15. *Assume $q > p - 1$ and $0 \in \partial\Omega$. Then there exists $c_{12} > 0$ depending on p, q and Ω such that for any positive solutions u_1, $u_2 \in C(\Omega \cup ((\partial\Omega \setminus \{0\}) \cap B_{2\delta_1})) \cap C^1(\Omega)$ of (1.1) in Ω, vanishing on $(\partial\Omega \setminus \{0\}) \cap B_{2\delta_1}$, there holds*

$$\sup\left\{\frac{u_1(y)}{u_2(y)} : y \in B_r \setminus B_{\frac{r}{2}}\right\} \leq c_{12} \inf\left\{\frac{u_1(y)}{u_2(y)} : y \in B_r \setminus B_{\frac{r}{2}}\right\}. \tag{2.51}$$

3 Boundary singularities

3.1 Strongly singular solutions

In this section we consider the equation (1.1) in $\mathbb{R}^N_+$. We denote by $(r,\sigma)\in\mathbb{R}_+\times S^{N-1}$ the spherical coordinates in $\mathbb{R}^N$ and

$$S^{N-1}_+=\left\{(\sin\phi\sigma',\cos\phi):\sigma'\in S^{N-2},\phi\in[0,\frac{\pi}{2})\right\}.$$

If $v(x)=r^{-\beta}\omega(\sigma)$ satisfies (1.1) in $\mathbb{R}^N_+$ and vanishes on $\partial\mathbb{R}^N_+\setminus\{0\}$, then $\beta=\beta_q$ and ω is a solution of

$$\begin{aligned}-div'\left(\left(\beta_q^2\omega^2+|\nabla'\omega|^2\right)^{\frac{p-2}{2}}\nabla'\omega\right)-\beta_q\Lambda_{\beta_q}\left(\beta_q^2\omega^2+|\nabla'\omega|^2\right)^{\frac{p-2}{2}}\omega&\\ +\left(\beta_q^2\omega^2+|\nabla'\omega|^2\right)^{\frac{q}{2}}=0\quad\text{in }S^{N-1}_+&\\ \omega=0\quad\text{on }\partial S^{N-1}_+.&\end{aligned}\tag{3.1}$$

where β_q and Λ_{β_q} have been defined in (1.10). We denote by $(\beta_*,\psi_*)\in\mathbb{R}^*_+\times C^2(\overline{S}^{N-1}_+)$ the unique couple such $\max\psi_*=1$ with the property that the function $(r,\sigma)\mapsto r^{-\beta_*}\psi_*(\sigma)$ is positive, p-harmonic in $\mathbb{R}^N_+$ and vanishes on $\partial\mathbb{R}^N_+\setminus\{0\}$. Then $\psi_*=\psi$ satisfies

$$\begin{aligned}-div'\left(\left(\beta_*^2\psi^2+|\nabla'\psi|^2\right)^{\frac{p-2}{2}}\nabla'\psi\right)-\beta_*\Lambda_{\beta_*}\left(\beta_*^2\psi^2+|\nabla'\psi|^2\right)^{\frac{p-2}{2}}\psi=0\quad\text{in }S^{N-1}_+&\\ \psi=0\quad\text{on }\partial S^{N-1}_+.&\end{aligned}\tag{3.2}$$

Since the function ψ_* is unique it depends only on the azimuthal variable $\theta_{N-1}=\cos^{-1}(\frac{x_N}{|x|})$ (see Appendix II). Our first result is the following

Theorem 3.1. *If $q\geq q_*$, or equivalently $\beta_q\leq\beta_*$, there exists no positive solution to problem* (3.1).

Proof. Suppose such a solution ω exists and put $\theta=\beta_q/\beta_*$, then $0<\theta\leq 1$. Set $\eta=\psi^\theta$, where ψ is a positive solution of (3.2), and define the operator $\mathcal{T}$ by

$$\begin{aligned}\mathcal{T}(\eta)=-div'\left(\left(\beta_q^2\eta^2+|\nabla'\eta|^2\right)^{\frac{p-2}{2}}\nabla'\eta\right)-\beta_q\Lambda_{\beta_q}\left(\beta_q^2\eta^2+|\nabla'\eta|^2\right)^{\frac{p-2}{2}}\eta&\\ +\left(\beta_q^2\eta^2+|\nabla'\eta|^2\right)^{\frac{q}{2}}.&\end{aligned}\tag{3.3}$$

Since $\nabla\eta=\theta\psi^{\theta-1}\nabla\psi$,

$$\left(\beta_q^2\eta^2+|\nabla'\eta|^2\right)^{\frac{p-2}{2}}=\theta^{p-2}\psi^{(\theta-1)(p-2)}\left(\beta_*^2\psi^2+|\nabla'\psi|^2\right)^{\frac{p-2}{2}},$$

$$\left(\beta_q^2\eta^2+|\nabla'\eta|^2\right)^{\frac{p-2}{2}}\nabla'\eta=\theta^{p-1}\psi^{(\theta-1)(p-1)}\left(\beta_*^2\psi^2+|\nabla'\psi|^2\right)^{\frac{p-2}{2}}\nabla'\psi,$$

therefore

$$\begin{aligned}\mathcal{T}(\eta)=&-\theta^{p-1}\psi^{(\theta-1)(p-1)}\mathrm{div}'\left(\left(\beta_*^2\psi^2+|\nabla'\psi|^2\right)^{\frac{p-2}{2}}\nabla'\psi\right)\\ &-\theta^{p-1}(\theta-1)(p-1)\psi^{(\theta-1)(p-1)-1}\left(\beta_*^2\psi^2+|\nabla'\psi|^2\right)^{\frac{p-2}{2}}|\nabla'\psi|^2\\ &-\beta_q\Lambda_{\beta_q}\theta^{p-2}\psi^{(\theta-1)(p-1)}\left(\beta_*^2\psi^2+|\nabla'\psi|^2\right)^{\frac{p-2}{2}}\psi+\theta^q\psi^{(\theta-1)q}\left(\beta_*^2\psi^2+|\nabla'\psi|^2\right)^{\frac{q}{2}}.\end{aligned}$$

But $\beta_q\Lambda_{\beta_q}\theta^{p-2}=\beta_*\Lambda_{\beta_q}\theta^{p-1}\leq\beta_*\Lambda_{\beta_*}\theta^{p-1}$ since $\beta_q\leq\beta_*$. Using (3.2), we see that $\mathcal{T}(\eta)\geq 0$. Because Hopf Lemma is valid, there holds $\partial_{\mathbf{n}}\psi<0$ on ∂S_+^{N-1}. Since ω is C^1 in $\overline{S_+^{N-1}}$ and ψ is defined up to an homothety, there exists a smallest function ψ such that $\eta\geq\omega$, and the graphs of η and ω over $\overline{S_+^{N-1}}$ are tangent, either at some $\alpha\in S_+^{N-1}$, or only at a point $\alpha\in\partial S_+^{N-1}$. We put $w=\eta-\omega$. Then

$$\mathcal{T}(\eta)=\mathcal{T}(\eta)-\mathcal{T}(\omega)=\Phi(1)-\Phi(0), \tag{3.4}$$

where $\Phi(t)=\mathcal{T}(\omega_t)$ with $\omega_t=\omega+tw$.

We use local coordinates $(\sigma_1,...,\sigma_{N-1})$ on S^{N-1} near α. We denote by $g=(g_{ij})$ the metric tensor on S^{N-1} and by g^{jk} its contravariant components. Then, for any $\varphi\in C^1(S^{N-1})$,

$$|\nabla\varphi|^2=\sum_{j,k}g^{jk}\frac{\partial\varphi}{\partial\sigma_j}\frac{\partial\varphi}{\partial\sigma_k}=\langle\nabla\varphi,\nabla\varphi\rangle_g.$$

If $X=(X^1,...X^d)\in C^1(TS^{N-1})$ is a vector field, we lower indices by setting $X^\ell=\sum_i g^{\ell i}X_i$ and define the divergence of X by

$$div'_gX=\frac{1}{\sqrt{|g|}}\sum_\ell\frac{\partial}{\partial\sigma_\ell}\left(\sqrt{|g|}X^\ell\right)=\frac{1}{\sqrt{|g|}}\sum_{\ell,i}\frac{\partial}{\partial\sigma_\ell}\left(\sqrt{|g|}g^{\ell i}X_i\right).$$

We write $\Phi(t)=\Phi_1(t)+\Phi_2(t)+\Phi_3(t)$ where

$$\Phi_1(t)=-\beta_q\Lambda_{\beta_q}\left(\beta_q^2\omega_t^2+|\nabla'\omega_t|^2\right)^{\frac{p-2}{2}}\omega_t\ ,\quad\Phi_2(t)=\left(\beta_q^2\omega_t^2+|\nabla'\omega_t|^2\right)^{\frac{q}{2}}$$

and

$$\Phi_3(t)=-\mathrm{div}'\left(\left(\mathrm{\beta_q^2\omega_t^2}+|\nabla'\mathrm{\omega_t}|^2\right)^{\frac{\mathrm{p}-2}{2}}\nabla'\mathrm{\omega_t}\right).$$

Then

$$\Phi_1(1)-\Phi_1(0)=-\sum_j a_j\frac{\partial w}{\partial\sigma_j}-bw\ \text{ and }\ \Phi_2(1)-\Phi_2(0)=\sum_j c_j\frac{\partial w}{\partial\sigma_j}+dw,$$

where

$$b=\beta_q\Lambda_{\beta_q}\left(\beta_q^2{\omega_t}^2+|\nabla\omega_t|^2\right)^{\frac{p}{2}-2}\left((p-1)\beta_q^2\omega_t^2+|\nabla\omega_t|^2\right),$$

$$a_j=(p-2)\beta_q\Lambda_{\beta_q}\left(\beta_q^2{\omega_t}^2+|\nabla\omega_t|^2\right)^{\frac{p}{2}-2}\omega_t\sum_k g^{jk}\frac{\partial\omega_t}{\partial\sigma_k},$$

$$d=q\beta_q^2\left(\beta^2{\omega_t}^2+|\nabla\omega_t|^2\right)^{\frac{q}{2}-1}\omega_t,$$

and

$$c_j=q\left(\beta_q^2{\omega_t}^2+|\nabla\omega_t|^2\right)^{\frac{q}{2}-1}\sum_k g^{jk}\frac{\partial\omega_t}{\partial\sigma_k}.$$

Furthermore

$$\begin{aligned}\Phi_3(1)-\Phi_3(0)=&-(p-2)\mathrm{div}'\left(\left(\beta_q^2\omega_t^2+|\nabla'\omega_t|^2\right)^{\frac{p-4}{2}}\left(\beta_q^2\omega_t w+\langle\nabla'\omega_t,\nabla' w\rangle_g\right)\nabla'\omega_t\right)\\&-\mathrm{div}'\left(\left(\beta_q^2\omega_t^2+|\nabla'\omega_t|^2\right)^{\frac{p-2}{2}}\nabla' w\right).\end{aligned}$$

Therefore we can write $\Phi(1)-\Phi(0)$ under the form

$$\Phi(1)-\Phi(0)=-\mathrm{div}'(A\nabla' w)+\langle B,\nabla' w\rangle_g+Cw:=\mathcal{L}w \tag{3.5}$$

where

$$\begin{aligned}\langle AX,X\rangle_g&=\left(\beta_q^2\omega_t^2+|\nabla'\omega_t|^2\right)^{\frac{p-4}{2}}\left(p-2)\langle\nabla'\omega_t,X\rangle_g^2+|\nabla'\omega_t|^2|X|^2\right)\\&\geq\left(\beta_q^2\omega_t^2+|\nabla'\omega_t|^2\right)^{\frac{p-4}{2}}\min\{1,p-1\}|\nabla'\omega_t|^2|X|^2.\end{aligned}\tag{3.6}$$

and B and C can be computed from the previous expressions. It is important to notice that $\beta_q^2\omega_t^2+|\nabla'\omega_t|^2$ is bounded between two positive constants m_1 and m_2 in $\overline{S_+^{N-1}}$. Thus the operator $\mathcal{L}$ is uniformly elliptic with bounded coefficients. Since w is nonnegative and either at some point α, $\nabla' w(\alpha)=0$ and $w(\alpha)>0$, or at some boundary point α where $w(\alpha)=0$ and $\partial_{\mathbf{n}}w(\alpha)<0$, it follows from the strong maximum principle or Hopf Lemma (see [7]) that $w=0$, contradiction. □

Theorem 3.2. *Assume $q<q_*$ or equivalently $\beta_q>\beta_*$. There exists a unique positive solution ω_* to problem* (3.1).

Proof. Existence. It will follow from [4]. Indeed problem (3.1) can be written under the form

$$\begin{aligned}\mathbf{A}(\omega):=-div'\,\mathbf{a}(\omega,\nabla'\omega)&=\mathbf{B}(\omega,\nabla'\omega)&&\text{in } S_+^{N-1}\\ \omega&=0&&\text{on } \partial S_+^{N-1},\end{aligned}\tag{3.7}$$

where

$$\begin{aligned}\mathbf{a}(r,\xi)&=\left(\beta_q^2r^2+|\xi|^2\right)^{\frac{p-2}{2}}\xi,\\ \mathbf{B}(r,\xi)&=\beta_q\Lambda_{\beta_q}\left(\beta_q^2r^2+|\xi|^2\right)^{\frac{p-2}{2}}r-\left(\beta_q^2r^2+|\xi|^2\right)^{\frac{q}{2}}.\end{aligned}\tag{3.8}$$

The operator $\mathbf{A}$ is a Leray-Lions operator which satisfies the assumptions (1.6)-(1.8) of [4, Theorem 2.1], and the term $\mathbf{B}$ satisfies (1.9),(1.10) in the same article. Therefore the existence of a positive solution $\omega\in W_0^{1,p}(S_+^{N-1})\cap L^\infty(S_+^{N-1})$ is ensured whenever we can find a supersolution $\overline{\omega}\in W^{1,p}(S_+^{N-1})\cap L^\infty(S_+^{N-1})$ and a nontrivial subsolution $\underline{\omega}\in W^{1,p}(S_+^{N-1})$ of (3.7) such that

$$0\leq\underline{\omega}\leq\overline{\omega}\qquad\text{in } S_+^{N-1}.\tag{3.9}$$

First we note that $\eta=\eta_0$ is a supersolution if the positive constant η_0 is large enough. In order to find a subsolution, we set again $\eta=\psi^\theta$ with $\theta=\beta_q/\beta_*$ and ψ as in (3.2). Now $\theta>1$, thus $\eta\in W_0^{1,p}(S_+^{N-1})$. As above we have

$$\begin{aligned}\mathcal{T}(\eta)=&-\theta^{p-1}\psi^{(\theta-1)(p-1)}\mathrm{div}'\left(\left(\beta_*^2\psi^2+|\nabla'\psi|^2\right)^{\frac{p-2}{2}}\nabla'\psi\right)\\&-\theta^{p-1}(\theta-1)(p-1)\psi^{(\theta-1)(p-1)-1}\left(\beta_*^2\psi^2+|\nabla'\psi|^2\right)^{\frac{p-2}{2}}|\nabla'\psi|^2\\&-\beta_q\Lambda_{\beta_q}\theta^{p-2}\psi^{(\theta-1)(p-1)}\left(\beta_*^2\psi^2+|\nabla'\psi|^2\right)^{\frac{p-2}{2}}\psi+\theta^q\psi^{(\theta-1)q}\left(\beta_*^2\psi^2+|\nabla'\psi|^2\right)^{\frac{q}{2}}.\end{aligned}$$

Now $\beta_q\Lambda_{\beta_q}\theta^{p-2}=\beta_*\Lambda_{\beta_q}\theta^{p-1}=\beta_*(\Lambda_{\beta_q}-\Lambda_{\beta_*})\theta^{p-1}+\beta_*\Lambda_{\beta_*}\theta^{p-1}$ and $\Lambda_{\beta_q}-\Lambda_{\beta_*}=(\beta_q-\beta_*)(p-1)=\beta_*(p-1)(\theta-1)$, hence

$$\begin{aligned}
\mathcal{T}(\eta)=&-\theta^{p-1}\psi^{(\theta-1)(p-1)}\mathrm{div}'\left(\left(\beta_*^2\psi^2+|\nabla'\psi|^2\right)^{\frac{p-2}{2}}\nabla'\psi\right)\\
&-\theta^{p-1}(\theta-1)(p-1)\psi^{(\theta-1)(p-1)-1}\left(\beta_*^2\psi^2+|\nabla'\psi|^2\right)^{\frac{p-2}{2}}|\nabla'\psi|^2\\
&-\beta_*(\Lambda_{\beta_q}-\Lambda_{\beta_*})\theta^{p-1}\psi^{(\theta-1)(p-1)}\left(\beta_*^2\psi^2+|\nabla'\psi|^2\right)^{\frac{p-2}{2}}\psi\\
&-\beta_*\Lambda_{\beta_*}\theta^{p-1}\psi^{(\theta-1)(p-1)}\left(\beta_*^2\psi^2+|\nabla'\psi|^2\right)^{\frac{p-2}{2}}\psi+\theta^q\psi^{(\theta-1)q}\left(\beta_*^2\psi^2+|\nabla'\psi|^2\right)^{\frac{q}{2}}.
\end{aligned}$$

Using the equation satisfied by ψ yields to the relation

$$\begin{aligned}
\mathcal{T}(\eta)=&-\theta^{p-1}(\theta-1)(p-1)\psi^{(\theta-1)(p-1)-1}\left(\beta_*^2\psi^2+|\nabla'\psi|^2\right)^{\frac{p-2}{2}}|\nabla'\psi|^2\\
&-\beta_*^2(p-1)(\theta-1)\theta^{p-1}\psi^{(\theta-1)(p-1)-1}\left(\beta_*^2\psi^2+|\nabla'\psi|^2\right)^{\frac{p-2}{2}}\psi^2\\
&+\theta^q\psi^{(\theta-1)q}\left(\beta_*^2\psi^2+|\nabla'\psi|^2\right)^{\frac{q}{2}}\\
=&-\theta^{p-1}(\theta-1)(p-1)\psi^{(\theta-1)(p-1)-1}\left(\beta_*^2\psi^2+|\nabla'\psi|^2\right)^{\frac{p}{2}}\\
&+\theta^q\psi^{(\theta-1)q}\left(\beta_*^2\psi^2+|\nabla'\psi|^2\right)^{\frac{q}{2}}.
\end{aligned}$$

If we replace $\eta:=\eta_1=\psi^\theta$ by $\eta:=\eta_m=(m\psi)^\theta$ in the above computation, the inequality $\mathcal{T}\eta_m)\leq 0$ will be true provided

$$m^{\theta(q+1-p)}\psi^{(\theta-1)(q+1-p)+1}\leq\theta^{p-1-q}(\theta-1)(p-1)\left(\beta_*^2\psi^2+|\nabla'\psi|^2\right)^{\frac{p-q}{2}},$$

which is satisfied if we choose m small enough so that $(m\psi)^\theta\leq\eta_0$ and satisfying

$$m^{\theta(q+1-p)}\leq\beta_*^{(\theta-1)(q+1-p)+1}\theta^{p-1-q}(\theta-1)(p-1)\frac{\min_{x\in S_+^{N-1}}\left(\beta_*^2\psi^2+|\nabla'\psi|^2\right)^{\frac{p-q}{2}}}{\max_{x\in S_+^{N-1}}\psi^{(\theta-1)(q+1-p)+1}}.$$

Therefore $0<\eta_m\leq\eta_0$ and standard regularity implies that the solution ω is C^1 in $\overline{S}_+^{N-1}$. Actually ω is C^∞ since the operator is not degenerate.

Uniqueness. We use the tangency method developed in the proof of Theorem 3.1. Assume ω_1 and ω_2 are two positive solutions of (3.2), then they are positive in S_+^{N-1} and $\partial_{\mathbf{n}}\omega_i<0$ on ∂S_+^{N-1}. Either the ω_i are ordered and $\omega_1\leq\omega_2$, or their graphs intersect. In any case we can define

$$\tau=\inf\{s>1:s\omega_1\geq\omega_2\}.$$

We set $\omega^*=\tau\omega_1$. Then either the graphs of ω_2 and ω^* are tangent at some interior point α, or they are not tangent in S_+^{N-1}, $\partial_{\mathbf{n}}\omega^*\leq\partial_{\mathbf{n}}\omega_2<0$ on ∂S_+^{N-1} and there exists $\alpha\in\partial S_+^{N-1}$ such that $\partial_{\mathbf{n}}\omega^*(\alpha)=\partial_{\mathbf{n}}\omega_2(\alpha)<0$. Furthermore $\mathcal{T}(\omega^*)\geq 0$. If we set $w=\omega^*-\omega_2$, then, as in Theorem 3.1,

$$-\mathrm{div}'(\tilde{A}\nabla'w)+\langle\tilde{B},\nabla'w\rangle_g+\tilde{C}w=\tilde{\mathcal{L}}w\geq 0$$

where

$$\begin{aligned}\langle \tilde{A}X, X\rangle_g &= \left(\beta_q^2\omega_t^2 + |\nabla'\omega_t|^2\right)^{\frac{p-4}{2}}\left((p-2)\langle\nabla'\omega_t, X\rangle_g^2 + |\nabla'\omega_t|^2|X|^2\right)\\ &\geq \left(\beta_q^2\omega_t^2 + |\nabla'\omega_t|^2\right)^{\frac{p-4}{2}}\min\{1, p-1\}|\nabla'\omega_t|^2|X|^2,\end{aligned} \tag{3.10}$$

in which $\omega_t = \omega_2 + t(\omega^* - \omega_2)$ and $t \in (0,1)$ is obtained by applying the mean value theorem and $\tilde{B}$ and $\tilde{C}$ are defined accordingly. Since $\tilde{\mathcal{L}}$ is uniformly elliptic and has bounded coefficients, it follows from the strong maximum principle that $w = 0$. Thus $\omega^* = \tau\omega_1 = \omega_2$ and $\tau = 1$ from the equation. This ends the proof. $\square$

3.2 Removable boundary singularities

The following is the basic result for removability of isolated singularities. It is valid in the general case, but with a local geometric constraint.

Theorem 3.3. *Assume $q^* \leq q < p \leq N$, Ω is a C^2 bounded domain with $0 \in \partial\Omega$, such that $\Omega \cap B_\delta = B_\delta^+$ for some $\delta > 0$. If $u \in C^1(\overline{\Omega} \setminus \{0\})$ is a nonnegative solution of* (1.1) *in Ω which vanishes on $\partial\Omega \setminus \{0\}$, then it is identically* 0.

Proof. Step 1: Assume $\Omega \subset \mathbb{R}_+^N$. For $\epsilon > 0$, we set $\Omega'_\epsilon = \Omega \cap \overline{B_\epsilon^c}$ and $H_\epsilon = \mathbb{R}_+^N \cap \overline{B_\epsilon^c}$. For $k, n \in \mathbb{N}_*$, $n \geq \operatorname{diam}(\Omega)$, we denote by $v_{k,n,\epsilon}$ ($n \in \mathbb{N}_*$) the solution of the problem

$$\begin{aligned}-\Delta_p v + |\nabla v|^q &= 0 && \text{in } H_\epsilon \cap B_n\\ v &= k\chi_{_{\mathbb{R}_+^N \cap \partial B_\epsilon}} && \text{on } \partial(H_\epsilon \cap B_n).\end{aligned} \tag{3.11}$$

If $k > c_2\epsilon^{\frac{q-p}{q+1-p}}$ for a suitable $c_2 = c_2(p,q) > 0$ (see Lemma 2.6), then $v_{k,n,\epsilon} \geq u$ in Ω'_ϵ. Moreover there holds $v_{k,n,\epsilon} \leq v_{k',n',\epsilon}$ for $n \leq n'$ and $k \leq k'$. Furthermore the function

$$U_{\epsilon,n}(x) = c_2\left((|x| - \epsilon)^{\frac{q-p}{q+1-p}} - (n-\epsilon)^{\frac{q-p}{q+1-p}}\right)$$

is a supersolution in $B_n \setminus B_\epsilon$, and there holds $v_{k,n,\epsilon} \leq U_{\epsilon,n}$. By monotonicity and standard a priori estimate, we obtain that $v_{k,n,\epsilon} \to v_\epsilon$ when $n, k \to \infty$ and that the function $v = v_\epsilon$ is solution of

$$\begin{aligned}-\Delta_p v + |\nabla v|^q &= 0 && \text{in } H_\epsilon\\ \lim_{|x|\to\epsilon} v(x) &= \infty &&\\ v &= 0 && \text{on } \partial\mathbb{R}_+^N \cap \overline{B_\epsilon^c}.\end{aligned} \tag{3.12}$$

Furthermore

$$u(x) \leq v_\epsilon(x) \leq c_2(|x| - \epsilon)^{\frac{q-p}{q+1-p}} \quad \text{in } \Omega'_\epsilon. \tag{3.13}$$

The function v_ϵ may not be unique, however it is the minimal solution of the above problem since the $v_{k,n,\epsilon}$ is unique, and monotonicity in n and k holds. Actually, $v_\epsilon \leq v_{\epsilon'}$ if $0 \leq \epsilon \leq \epsilon'$. For $\ell > 0$, we recall that the transformation $v \mapsto T_\ell[v]$ defined by (2.41) leaves equation (1.1) invariant. As a consequence of the uniqueness of the approximations we have $T_\ell[v_{k,n,\epsilon}] = v_{\ell^{\frac{p-q}{q+1-p}}k, \ell^{-1}n, \ell^{-1}\epsilon}$, which implies

$$T_\ell[v_\epsilon] = v_{\ell^{-1}\epsilon}. \tag{3.14}$$

Letting $\epsilon \to 0$, we derive from the monotonicity with respect to ϵ and standard $C^{1,\alpha}$ estimates, that the following identity holds:

$$T_\ell[v_0] = v_0 \qquad \forall \ell > 0. \tag{3.15}$$

The function v_0 is a positive and separable solution of (1.1) in $\mathbb{R}^N_+$ which vanishes on $\partial\Omega\setminus\{0\}$. It follows from Theorem 3.1 that $v_0 = 0$, and so is u.

Step 2: The general case. We assume that $\Omega \cap B_\delta \subset \mathbb{R}^N_+$ and we denote by M the maximum of u on $\partial B_\delta \cap \Omega$. Then the function $(u-M)_+$ is a subsolution of (1.1) in $\Omega\cap B_\delta$ which vanishes on $\partial\Omega\cap B_\delta\setminus\{0\}$. By Step 1, it is dominated by v_0, which ends the proof. $\square$

Remark. The previous result is valid if u is a subsolution with the same regularity. If u is no longer assumed to be nonnegative, only u^+ vanishes. Furthermore, the regularity of the boundary has not been used, but only the fact that Ω is locally contained into a half space to the boundary of which 0 belongs.

Remark. If no geometric assumption is made on $\partial\Omega$, we can prove that $u(x) = o(|x|^{-\beta_q})$ near 0. The next result shows that the removability holds if $q > q_*$.

Theorem 3.4. *Assume $q^* < q < p \leq N$ and Ω is a C^2 bounded domain with $0 \in \partial\Omega$. If u is a nonnegative solution of (1.1) in Ω which belongs to $C^1(\overline{\Omega} \setminus \{0\})$ and vanishes on $\partial\Omega \setminus \{0\}$, it is identically* 0.

Proof. As it is proved in [12], for any smooth subdomain $S \subset S^{N-1}$, there exists a unique $\beta_{*\,s} > 0$ and $\psi^*_s > 0$, unique up to an homothety, such that $x \mapsto |x|^{-\beta_{*\,s}}\,\psi^*_s(|x|^{-1}\,x)$ is p harmonic in the cone $C_S = \{x \in \mathbb{R}^N \setminus \{0\} : |x|^{-1}\,x \in S\}$ and ψ^*_s satisfies

$$\begin{aligned} -div'\left(\left(\beta^2_{*\,s}\psi^{*\,2}_s + |\nabla'\psi^*_s|^2\right)^{\frac{p-2}{2}}\nabla'\psi^*_s\right) - \beta_{*\,s}\Lambda_{\beta_{*\,s}}\left(\beta^2_{*\,s}\psi^{*\,2} + |\nabla'\psi^*_s|^2\right)^{\frac{p-2}{2}}\psi^*_s = 0 \qquad &\text{in } S \\ \psi^*_s = 0 \qquad &\text{on } \partial S, \end{aligned} \tag{3.16}$$

Furthermore $S \subset \tilde{S} \subset S^{N-1}$ implies $\beta_{*\,\tilde{s}} \leq \beta_{*\,s}$. Using the system of spherical coordinates defined in (6.5) in Appendix II, for $\epsilon > 0$ we denote by $S := S_\epsilon$ the spherical shell with vertex the north pole N and latitude angle $\theta_{N-1} \in [0, \frac{\pi}{2} + \epsilon]$. Because of uniqueness of $\beta_{*\,s}$, $\beta_{*\,s_\epsilon} \uparrow \beta_*$ as $\epsilon \to 0$. Therefore, if $q > q_*$, or equivalently $\beta_q < \beta_*$, there exists $\delta, \epsilon > 0$ such that $\Omega \cap B_\delta \subset C_{S_\epsilon} \cap B_\delta$ and $\beta_q < \beta_{*\,s_\epsilon}$. Since Theorem 3.1 is valid if S^{N-1}_+ is replaced by S_ϵ and $\beta_q < \beta_{*\,s_\epsilon}$ it follows that $u = 0$ as in the proof of Theorem 3.3, Steps 1 and 2. $\square$

The next result, valid in the case $p = N$, is based upon the conformal invariance of the N-Laplacian. In this case the exponent β_* corresponding to the first spherical N-harmonic eigenvalue is equal to 1 and the corresponding spherical N-harmonic eigenfunction in S^{N-1}_+ is $x_N/\,|x|^2$.

Theorem 3.5. *Assume $N - \frac{1}{2} \leq q < N$, Ω is a bounded domain and $0 \in \partial\Omega$ is such that there exists a ball $B \subset \Omega^c$ to the boundary of which 0 belongs. If u is a nonnegative solution of*

$$-\Delta_N u + |\nabla u|^q = 0 \qquad \textit{in } \Omega, \tag{3.17}$$

which belongs to $C(\overline{\Omega} \setminus \{0\}) \cap W^{1,N}_0(\Omega \setminus \overline{B}_\epsilon(0))$ for any $\epsilon > 0$, it is identically 0.

Proof. We assume that the inward normal unit vector to $\partial\Omega$ at 0 is $\mathbf{e}_N = (0,0,...,1)$ and that the ball $B = B_{\frac{1}{2}}(a)$ of center $a = -\frac{1}{2}\mathbf{e}_N$ and radius $\frac{1}{2}$ touches $\partial\Omega$ at 0 and is exterior to Ω (this can be assumed up to a rotation and a dilation). This is the consequence of the exterior sphere condition at the point 0. It is always valid if $\partial\Omega$ is C^2. We denote by $\mathcal{I}_\omega$ the inversion of center $\omega = -\mathbf{e}_N$ and power 1, i.e. $\mathcal{I}_\omega(x) = \omega + \frac{x-\omega}{|x-\omega|^2}$. Under this transformation, the complement of the ball $B_{\frac{1}{2}}(a)$, which contains Ω, is transformed into the half space $\mathbb{R}^N_-$ which contains the image $\tilde{\Omega}$ of Ω. Since u satisfies (3.17), $\tilde{u} = u\circ\mathcal{I}_\omega$ satisfies

$$-\Delta_N\tilde{u} + |x-\omega|^{2(q-N)}|\nabla\tilde{u}|^q = 0 \qquad \text{in } \tilde{\Omega}. \tag{3.18}$$

Furthermore since $0 = \mathcal{I}_\omega(0)$ and $\mathcal{I}_\omega$ is a diffeomorphism, $\tilde{u} \in C(\overline{\tilde{\Omega}}\setminus\{0\}) \cap C^1(\tilde{\Omega})$ and it vanishes on $\partial\tilde{\Omega}\setminus\{0\}$. Since $|x-\omega| \leq 1$ and $q < N$, $\tilde{u}$ is a subsolution for (3.17) in $\tilde{G}$. By Theorem 3.4, $\tilde{u} = 0$. $\square$

3.3 Weakly singular solutions

The main result of this section is the following existence and uniqueness result concerning solutions of (1.1) with a boundary weak singularity. We recall that ψ_* is unique positive solution of (1.11) such that $\sup\psi_* = 1$. Our first result is valid for any $1 < p \leq N$ but it needs a geometric constraint on Ω.

Theorem 3.6. *Let $p-1 < q < q_* < p \leq N$ and $\Omega \subset \mathbb{R}^N_+$ be a bounded C^2 domain such that $0 \in \partial\Omega$. Assume that there exists $\delta > 0$ such that $\Omega_\delta := \Omega\cap B_\delta = B^+_\delta$. Then for any $k > 0$ there exists a unique positive solution $u := u_k$ of (1.1) in Ω, which belongs to $C^1(\overline{\Omega}\setminus\{0\})$, vanishes on $\partial\Omega\setminus\{0\}$ and satisfies*

$$\lim_{x\to 0}\frac{u_k(x)}{\Psi_*(x)} = k \tag{3.19}$$

in the C^1-topology of S^{N-1}_+, where

$$\Psi_*(x) = |x|^{-\beta_*}\psi_*(|x|^{-1}x).$$

The proof of this theorem is long and difficult and requires a certain number of intermediate results.

Lemma 3.7. *Let the assumptions on p, q and Ω of Theorem 3.6 be satisfied. There exists a unique positive p-harmonic function Φ_* in Ω, which is continuous in $\overline{\Omega}\setminus\{0\}$, vanishes on $\partial\Omega\setminus\{0\}$ and satisfies*

$$\lim_{x\to 0}\frac{\Phi_*(x)}{\Psi_*(x)} = 1. \tag{3.20}$$

Proof. For $0 < \epsilon < \delta$ let v_ϵ be the unique nonnegative p-harmonic function in $\Omega\setminus\overline{B^+_\epsilon}$ which is continuous in $\overline{\Omega}\setminus B^+_\epsilon$, vanishes on $\partial\Omega\setminus B_\epsilon$ and achieves the value Ψ_* on $\partial B_\epsilon\cap\Omega$. Since $\Omega\subset\mathbb{R}^N_+$, $v_\epsilon \leq \Psi_*$ in $\Omega\setminus B^+_\epsilon$. Hence inequalities $0 < \epsilon < \epsilon' \leq \delta$ imply $v_\epsilon \leq v_{\epsilon'}$ in $\Omega\setminus\overline{B^+_{\epsilon'}}$. Because $\Psi_* \leq \delta^{-\beta_*}$, there holds

$$v_\epsilon + \delta^{-\beta_*} \geq \Psi_*, \tag{3.21}$$

in $\Omega\setminus B^+_\delta$. Since v_ϵ and Ψ_* coincide on ∂B^+_ϵ and vanish on $\partial\mathbb{R}^N_+\cap(B^+_\delta\setminus B^+_\epsilon)$, (3.21) holds also in $B^+_\delta\setminus B^+_\epsilon$. Because $v_\epsilon \geq 0$ there holds

$$(\Psi_* - \delta^{-\beta_*})_+ \leq v_\epsilon \leq \Psi_* \quad \text{ in } \Omega\setminus B^+_\epsilon. \tag{3.22}$$

By a standard regularity result v_ϵ converges to a function Φ_* continuous in $\overline{\Omega}\setminus\{0\}$, p-harmonic in Ω such that

$$(\Psi_* - \delta^{-\beta_*})_+ \leq \Phi_* \leq \Psi_*$$

in Ω. Therefore (3.20) holds provided $\frac{x}{|x|}$ remains in a compact subset of S_+^{N-1}. Let us define a function $\tilde{\phi}_*$ by $\tilde{\phi}_*(x) = |x|^{\beta_*}\,\Phi_*(x)$, then $\tilde{\phi}_*(r,\sigma) \leq \psi_*(\sigma)$ where $r = |x|$ and $\sigma = \frac{x}{|x|} \in S_+^{N-1}$. By standard $C^{1,\alpha}$ estimates, $\tilde{\phi}_*(r,.)$ is relatively compact in the $C(\overline{S_+^{N-1}})$-topology. Therefore the convergence of $\frac{\Phi_*(x)}{\Psi_*(x)}$ to 1 when x to 0 holds not only when $\frac{x}{|x|}$ remains in a compact subset of S_+^{N-1}, but uniformly on S_+^{N-1}, which implies (3.20). Uniqueness follows classically by (3.20) and the maximum principle. □

Lemma 3.8. *Let the assumptions on p, q and Ω of Theorem 3.6 be satisfied. If for some $k > 0$ there exists a solution u_k of (1.1) in Ω, which belongs to $C^1(\overline{\Omega}\setminus\{0\})$, vanishes on $\partial\Omega\setminus\{0\}$ and satisfies (3.19), then for any $k > 0$ there exists such a solution.*

Proof. We notice that for any $c < 1$ (resp $c > 1$), cu_k is a subsolution (resp. supersolution) of (1.1) in Ω. Let Φ_* be as in Lemma 3.7. If $c < 1$, the function $ck\Phi_*$ is a supersolution of (1.1) which vanishes on $\partial\Omega\setminus\{0\}$. Furthermore

$$\lim_{x\to 0}\frac{cu_k(x)}{\Psi_*(x)} = ck = \lim_{x\to 0}\frac{ck\Phi_*(x)}{\Psi_*(x)}.$$

Then there exists a solution u_{ck} of (1.1) in Ω which satisfies $cu_k \leq u_{ck} \leq ck\Phi_*$. If $c > 1$, we set $u^* := T_{c^\theta}[u_k]$, which means $u^*(x) = c^{\beta_q\theta}u_k(c^\theta\, x)$ with $\theta = (\beta_q - \beta_*)^{-1}$. Then u^* is a solution of (1.1) in $\Omega^{c^\theta} = \frac{1}{c^\theta}\Omega$. In particular, u^* satisfies the equation in $B^+_{\frac{\delta}{c^\theta}}(0)$. Since $c^\theta > 1$, $B^+_{\frac{\delta}{c^\theta}}(0) \subset B^+_\delta(0)$. Put $m = \max\{u^* : x \in \partial B^+_{\frac{\delta}{c^\theta}}(0)\}$. The function $(u^* - m)_+$, extended by 0 outside $B^+_{\frac{\delta}{c^\theta}}(0)$, is a subsolution of (1.1) in Ω. Furthermore it satisfies

$$\lim_{x\to 0}\frac{(u^* - m)_+(x)}{\Psi_*(x)} = ck,$$

uniformly on any compact subset of S_+^{N-1}. Therefore there exists a solution u_{ck} of (1.1) in Ω which satisfies $(u^* - m)_+ \leq u_{ck} \leq ck\Phi_*$, and in particular it vanishes on $\partial\Omega\setminus\{0\}$ and belongs to $C^1(\overline{\Omega}\setminus\{0\})$. By [13], u_{ck} is positive in Ω. Thus u_{ck} belongs to $C^{1,\alpha}(\overline{B^+_\delta}(0)\setminus\{0\})$ and satisfies

$$|x|^{\beta_*}\,|u_{ck}(x)| + |x|^{1+\beta_*}\,|\nabla u_{ck}(x)| + |x|^{1+\beta_*+\alpha}\sup_{\substack{|y|\leq|x|\\ x\neq y}}\frac{|\nabla u_{ck}(x) - \nabla u_{ck}(y)|}{|x-y|^\alpha} \leq M$$

by (2.11). Therefore the set of functions $\{r^{\beta_*+1}\nabla u_{ck}(r,.)\}_{r>0}$ is uniformly relatively compact in the topology of uniform convergence on $\overline{S}_+^{N-1}$. Since it converges to $ck\nabla'\psi_*$ uniformly on compact subsets of S_+^{N-1} as $r\to 0$, this convergence holds in $C(\overline{S_+^{N-1}})$. This implies

$$\lim_{x\to 0}\frac{u_{ck}(x)}{\Psi_*(x)} = ck. \tag{3.23}$$

□

The next Lemma is the keystone of our construction. Its proof is very delicate and needs several intermediate steps.

Lemma 3.9. *Under the assumptions of Theorem 3.6 there exists a real number R_0 such that* $0 < R_0 \leq \delta$ *and a positive subsolution $\tilde{u}$ of* (1.1) *in $B^+_{R_0}$ which is Lipschitz continuous in* $\overline{B^+_{R_0}} \setminus \{0\}$, *vanishes on* $\overline{B^+_{R_0}} \cap \partial\mathbb{R}^N_+ \setminus \{0\}$, *is smaller than Ψ_* and satisfies*

$$\lim_{x\to 0} \frac{\tilde{u}(x)}{\Psi_*(x)} = 1. \tag{3.24}$$

Proof. The construction of the function $\tilde{u}$. We look for a subsolution under the form $\tilde{u} = \Psi_* - w$ for a suitable nonnegative function w.

Step 1: reduction of the problem. We use spherical coordinates for a C^1 function $u : x \mapsto u(x) = u(r,\sigma)$, $r = |x|$, $\sigma = \frac{x}{|x|}$. Then $\nabla u = u_r\mathbf{e} + r^{-1}\nabla' u$ where $\mathbf{e} = |x|^{-1}x$, $|\nabla u|^2 = u_r^2 + r^{-2}|\nabla' u|^2$ and $|\nabla u|^q = \left(u_r^2 + r^{-2}|\nabla' u|^2\right)^{\frac{q}{2}}$. The expression of the p-Laplacian in spherical coordinates is

$$\begin{aligned}-\Delta_p u = &-\left(\left(u_r^2 + r^{-2}|\nabla' u|^2\right)^{\frac{p-2}{2}} u_r\right)_r - \frac{N-1}{r}\left(u_r^2 + r^{-2}|\nabla' u|^2\right)^{\frac{p-2}{2}} u_r \\ &- \frac{1}{r^2}div'\left(\left(u_r^2 + r^{-2}|\nabla' u|^2\right)^{\frac{p-2}{2}} \nabla' u\right).\end{aligned}$$

Put $v(t,\sigma) = r^{\beta_*}u(r,\sigma)$ with $t = \ln r \in (-\infty, \ln\delta]$, then v satisfies

$$\begin{aligned}&\mathcal{Q}[v] := \\ &-\left(\left((v_t-\beta_*v)^2 + |\nabla' v|^2\right)^{\frac{p-2}{2}}(v_t-\beta_*v)\right)_t - div'\left(\left((v_t-\beta_*v)^2 + |\nabla' v|^2\right)^{\frac{p-2}{2}}\nabla' v\right) \\ &\quad + \Lambda_{\beta_*}\left((v_t-\beta_*v)^2 + |\nabla' v|^2\right)^{\frac{p-2}{2}}(v_t-\beta_*v) + e^{\nu t}\left((v_t-\beta_*v)^2 + |\nabla' v|^2\right)^{\frac{q}{2}} = 0\end{aligned} \tag{3.25}$$

in $(-\infty,\ln\delta)\times S^{N-1}_+$ where $\nu = 1-(q+1-p)(\beta_*+1) = 1-\frac{\beta_*+1}{\beta_q+1} > 0$ and $\Lambda_{\beta_*} = \beta_*(p-1)+p-N$. Notice that ψ_* satisfies

$$-div'\left(\left(\beta_*^2\psi_*^2 + |\nabla'\psi_*|^2\right)^{\frac{p-2}{2}}\nabla'\psi_*\right) - \beta_*\Lambda_{\beta_*}\left(\beta_*^2\psi_*^2 + |\nabla'\psi_*|^2\right)^{\frac{p-2}{2}}\psi_* = 0, \tag{3.26}$$

hence it is a supersolution for (3.25). We look for a subsolution under the form

$$V(t,\sigma) = \psi_* - a(t)g(\psi_*)$$

where g is a continuous increasing function defined on $\mathbb{R}_+$, vanishing at 0 and smooth on $\mathbb{R}^*_+$ and $a(t) = e^{\gamma t}$ with $\gamma > 0$ to be chosen. Thus $a' = \gamma a$, $a'' = \gamma^2 a$, $V_t = -\gamma a g(\psi_*)$, $V_t - \beta_* V =$

$-\beta_*\psi_* + a(\beta_* - \gamma)g(\psi_*)$, $\nabla' V = (1 - ag'(\psi_*))\nabla'\psi_*$ and

$$\begin{aligned}(V_t - \beta_* V)^2 + |\nabla' V|^2 &= (-\beta_*\psi_* + a(\beta_* - \gamma)g(\psi_*))^2 + (1 - ag'(\psi_*))^2 |\nabla'\psi_*|^2 \\ &= \left(\beta_*^2\psi_*^2 + 2a\beta_*(\gamma - \beta_*)g(\psi_*)\psi_*\right) + (1 - 2ag'(\psi_*))\, |\nabla'\psi_*|^2 + O(a^2 \|g(\psi)\|_{C^1}) \\ &= \beta_*^2\psi_*^2 + |\nabla'\psi_*|^2 + 2a\left(\beta_*(\gamma - \beta_*)\psi_* g(\psi_*) - g'(\psi_*)\, |\nabla\psi_*|^2\right) + O(a^2 \|g(\psi_*)\|_{C^1}).\end{aligned}$$

Therefore

$$\begin{aligned}&\left((V_t - \beta_* V)^2 + |\nabla' V|^2\right)^{\frac{p-2}{2}} \\ &= \left(\beta_*^2\psi_*^2 + |\nabla'\psi_*|^2\right)^{\frac{p-2}{2}} \left[1 + (p-2)a\frac{\beta_*(\gamma - \beta_*)\psi_* g(\psi_*) - g'(\psi_*)\, |\nabla\psi_*|^2}{\beta_*^2\psi_*^2 + |\nabla'\psi_*|^2}\right] \\ &\qquad + O(a^2 \|g(\psi)\|_{C^1}),\end{aligned}$$

and

$$\begin{aligned}&e^{\nu t}\left((V_t - \beta_* V)^2 + |\nabla' V|^2\right)^{\frac{q}{2}} \\ &= e^{\nu t}\left(\beta_*^2\psi_*^2 + |\nabla'\psi_*|^2\right)^{\frac{q}{2}} \left[1 + qa\frac{\beta_*(\gamma - \beta_*)\psi_* g(\psi_*) - g'(\psi_*)\, |\nabla\psi_*|^2}{\beta_*^2\psi_*^2 + |\nabla'\psi_*|^2}\right] \\ &\qquad + O(e^{\nu t}a^2 \|g(\psi_*)\|_{C^1}),\end{aligned}$$

thus

$$\begin{aligned}&\left((V_t - \beta_* V)^2 + |\nabla' V|^2\right)^{\frac{p-2}{2}} (V_t - \beta_* V) \\ &\quad = -\beta_*\left(\beta_*^2\psi_*^2 + |\nabla'\psi_*|^2\right)^{\frac{p-2}{2}} \psi_* + a(\beta_* - \gamma)\left(\beta_*^2\psi_*^2 + |\nabla'\psi_*|^2\right)^{\frac{p-2}{2}} g(\psi_*) \\ &\qquad - a\beta_*(p-2)\frac{\beta_*(\gamma - \beta_*)\psi_* g(\psi_*) - g'(\psi_*)\, |\nabla\psi_*|^2}{(\beta_*^2\psi_*^2 + |\nabla'\psi_*|^2)^{\frac{4-p}{2}}}\psi_* + O(a^2 \|g(\psi_*)\|_{C^1}).\end{aligned}$$

Finally,

$$\begin{aligned}&-\left(\left((V_t - \beta_* V)^2 + |\nabla' V|^2\right)^{\frac{p-2}{2}} (V_t - \beta_* V)\right)_t \\ &= a\left[(\gamma^2 - \beta_*\gamma)\left(\beta_*^2\psi_*^2 + |\nabla'\psi_*|^2\right)^{\frac{p-2}{2}} g(\psi_*)\right. \\ &\left. + \beta_*(p-2)\frac{\beta_*(\gamma^2 - \beta_*\gamma)\psi_* g(\psi_*) - \gamma g'(\psi_*)\, |\nabla\psi_*|^2}{(\beta_*^2\psi_*^2 + |\nabla'\psi_*|^2)^{\frac{4-p}{2}}}\psi_*\right] + O(a^2 \|g(\psi_*)\|_{C^2}).\end{aligned} \tag{3.27}$$

Since

$$\begin{aligned}
&\left((V_t-\beta_*V)^2+|\nabla'V|^2\right)^{\frac{p-2}{2}}\nabla'V=\\
&\left(\beta_*^2\psi_*^2+|\nabla'\psi_*|^2\right)^{\frac{p-2}{2}}(1-ag'(\psi_*))\left[1+a(p-2)\frac{\beta_*(\gamma-\beta_*)\psi_*g(\psi_*)-g'(\psi_*)\left|\nabla\psi_*\right|^2}{\beta_*^2\psi_*^2+|\nabla'\psi_*|^2}\right]\nabla'\psi_*\\
&\qquad\qquad\qquad\qquad\qquad\qquad+O(a^2\left\|g(\psi_*)\right\|_{C^1})\\
&\qquad\qquad\qquad=\left(\beta_*^2\psi_*^2+|\nabla'\psi_*|^2\right)^{\frac{p-2}{2}}\nabla'\psi_*\\
&\qquad+a\left(\beta_*^2\psi_*^2+|\nabla'\psi_*|^2\right)^{\frac{p-2}{2}}\left[(p-2)\frac{\beta_*(\gamma-\beta_*)\psi_*g(\psi_*)-g'(\psi_*)\left|\nabla\psi_*\right|^2}{\beta_*^2\psi_*^2+|\nabla'\psi_*|^2}-g'(\psi_*)\right]\nabla'\psi_*\\
&\qquad\qquad\qquad\qquad\qquad\qquad+O(a^2\left\|g(\psi_*)\right\|_{C^1}),
\end{aligned}$$

we get similarly

$$\begin{aligned}
&-div'\left(\left((V_t-\beta_*V)^2+|\nabla'V|^2\right)^{\frac{p-2}{2}}\nabla'V\right)=-div'\left(\left(\beta_*^2\psi_*^2+|\nabla'\psi_*|^2\right)^{\frac{p-2}{2}}\nabla'\psi_*\right)\\
&-a\,div'\left(\left(\beta_*^2\psi_*^2+|\nabla'\psi_*|^2\right)^{\frac{p-2}{2}}\left[(p-2)\frac{\beta_*(\gamma-\beta_*)\psi_*g(\psi_*)-g'(\psi_*)\left|\nabla\psi_*\right|^2}{\beta_*^2\psi_*^2+|\nabla'\psi_*|^2}-g'(\psi_*)\right]\nabla'\psi_*\right)\\
&\qquad\qquad\qquad\qquad\qquad\qquad+O(a^2\left\|g(\psi_*)\right\|_{C^2}).
\end{aligned}\tag{3.28}$$

Noting that

$$-div'\left(\left(\beta_*^2\psi_*^2+|\nabla'\psi_*|^2\right)^{\frac{p-2}{2}}\nabla'\psi_*\right)\psi_*=\beta_*\Lambda_{\beta_*}\left(\beta_*^2\psi_*^2+|\nabla'\psi_*|^2\right)^{\frac{p-2}{2}}\psi_*,\tag{3.29}$$

we obtain

$$\begin{aligned}
&e^{-\gamma t}\mathcal{Q}[V]\\
&=\left[(\gamma^2-\beta_*\gamma)\left(\beta_*^2\psi_*^2+|\nabla'\psi_*|^2\right)^{\frac{p-2}{2}}g(\psi_*)+\beta_*(p-2)\frac{\beta_*(\gamma^2-\beta_*\gamma)\psi_*g(\psi_*)-\gamma g'(\psi_*)\left|\nabla\psi_*\right|^2}{\left(\beta_*^2\psi_*^2+|\nabla'\psi_*|^2\right)^{\frac{4-p}{2}}}\psi_*\right]\\
&-div'\left(\left(\beta_*^2\psi_*^2+|\nabla'\psi_*|^2\right)^{\frac{p-2}{2}}\left[(p-2)\frac{\beta_*(\gamma-\beta_*)\psi_*g(\psi_*)-g'(\psi_*)\left|\nabla\psi_*\right|^2}{\beta_*^2\psi_*^2+|\nabla'\psi_*|^2}-g'(\psi_*)\right]\nabla'\psi_*\right)\\
&-\Lambda_{\beta_*}\left((\gamma-\beta_*)\left(\beta_*^2\psi_*^2+|\nabla'\psi_*|^2\right)^{\frac{p-2}{2}}g(\psi_*)+\beta_*(p-2)\frac{\beta_*(\gamma-\beta_*)\psi_*g(\psi_*)-g'(\psi_*)\left|\nabla\psi_*\right|^2}{\left(\beta_*^2\psi_*^2+|\nabla'\psi_*|^2\right)^{\frac{4-p}{2}}}\psi_*\right)\\
&+e^{(\nu-\gamma)t}\left(\beta_*^2\psi_*^2+|\nabla'\psi_*|^2\right)^{\frac{q}{2}}\left[1+qa\frac{\beta_*(\gamma-\beta_*)\psi_*g(\psi_*)-g'(\psi_*)\left|\nabla\psi_*\right|^2}{\beta_*^2\psi_*^2+|\nabla'\psi_*|^2}\right]+O(a\left\|g(\psi_*)\right\|_{C^2}).
\end{aligned}\tag{3.30}$$

In this expression we have in particular

$$
\begin{aligned}
-div'\left(\left(\beta_*^2\psi_*^2+|\nabla'\psi_*|^2\right)^{\frac{p-2}{2}}\left[(p-2)\frac{\beta_*(\gamma-\beta_*)\psi_*g(\psi_*)-g'(\psi_*)\,|\nabla\psi_*|^2}{\beta_*^2\psi_*^2+|\nabla'\psi_*|^2}-g'(\psi_*)\right]\nabla'\psi_*\right)\\
&=(p-1)div'\left[g'(\psi_*)\left(\beta_*^2\psi_*^2+|\nabla'\psi_*|^2\right)^{\frac{p-2}{2}}\nabla\psi_*\right]\\
&-\beta_*div'\left(\left(\beta_*^2\psi_*^2+|\nabla'\psi_*|^2\right)^{\frac{p-4}{2}}\left[(p-2)\beta_*\psi_*g'(\psi_*)+(p-2)(\gamma-\beta_*)g(\psi_*)\right]\psi_*\right)\\
&=(p-1)g''(\psi_*)\left(\beta_*^2\psi_*^2+|\nabla'\psi_*|^2\right)^{\frac{p-2}{2}}|\nabla\psi_*|^2\\
&+(p-1)g'(\psi_*)div'\left(\left(\beta_*^2\psi_*^2+|\nabla'\psi_*|^2\right)^{\frac{p-2}{2}}\nabla\psi_*\right)\\
&-(p-2)\beta_*div'\left[\frac{\left((\gamma-\beta_*)g(\psi_*)\psi_*+\beta_*g'(\psi_*)\psi_*^2\right)}{\left(\beta_*^2\psi_*^2+|\nabla'\psi_*|^2\right)^{\frac{4-p}{2}}}\nabla'\psi_*\right].
\end{aligned}
\tag{3.31}
$$

Using the equation (3.26) satisfied by ψ_*, it infers that

$$
\begin{aligned}
-div'\left(\left(\beta_*^2\psi_*^2+|\nabla'\psi_*|^2\right)^{\frac{p-2}{2}}\left[(p-2)\frac{\beta_*(\gamma-\beta_*)\psi_*g(\psi_*)-g'(\psi_*)\,|\nabla\psi_*|^2}{\beta_*^2\psi_*^2+|\nabla'\psi_*|^2}-g'(\psi_*)\right]\nabla'\psi_*\right)\\
&=(p-1)\left(\beta_*^2\psi_*^2+|\nabla'\psi_*|^2\right)^{\frac{p-2}{2}}\left(g''(\psi_*)|\nabla'\psi_*|^2-\beta_*\Lambda_{\beta_*}g'(\psi_*)\psi_*\right)\\
&-(p-2)\beta_*div'\left[\frac{\left((\gamma-\beta_*)g(\psi_*)\psi_*+\beta_*g'(\psi_*)\psi_*^2\right)}{\left(\beta_*^2\psi_*^2+|\nabla'\psi_*|^2\right)^{\frac{4-p}{2}}}\nabla'\psi_*\right].
\end{aligned}
\tag{3.32}
$$

Plugging this identity into the expression (3.30), we obtain after some simplifications

$$
e^{-\gamma t}\mathcal{Q}[V]=\left(\beta_*^2\psi_*^2+|\nabla'\psi_*|^2\right)^{\frac{p-2}{2}}g(\psi_*)\mathcal{Q}_1[V]+e^{(\nu-\gamma)t}R[V]+O(a\,\|g(\psi_*)\|_{C^2}),\tag{3.33}
$$

where

$$
R[V]=e^{\nu t}\left(\beta_*^2\psi_*^2+|\nabla'\psi_*|^2\right)^{\frac{q}{2}}\left[1+q\frac{\beta_*(a'-\beta_*a)\psi_*g(\psi_*)-ag'(\psi_*)\,|\nabla\psi_*|^2}{\beta_*^2\psi_*^2+|\nabla'\psi_*|^2}\right],\tag{3.34}
$$

and

$$\begin{aligned}\mathcal{Q}_1[V] = (\gamma - \Lambda_{\beta_*})(\gamma - \beta_*) \left[1 + (p-2)\frac{\beta_*^2\psi_*^2}{\beta_*^2\psi_*^2 + |\nabla'\psi_*|^2}\right] - (p-1)\beta_*\Lambda_{\beta_*}\frac{\psi_* g'(\psi_*)}{g(\psi_*)} \\ + [(p-4)\beta_*\Lambda_{\beta_*}\psi_* - 2\Delta'\psi_*]\left(\gamma - \beta_*\left(1 - \frac{\psi_* g'(\psi_*)}{g(\psi_*)}\right)\right)\frac{\beta_*\psi_*}{\beta_*^2\psi_*^2 + |\nabla'\psi_*|^2} \\ -(p-2)\left[\frac{\psi_* g'(\psi_*)}{g(\psi_*)}\left((\beta_*+1)\gamma - \beta_*\Lambda_{\beta_*} + \beta_*\right) + \gamma - \beta_* + \beta_*\frac{\psi_*^2 g''(\psi_*)}{g(\psi_*)}\right]\frac{|\nabla'\psi_*|^2}{\beta_*^2\psi_*^2 + |\nabla'\psi_*|^2} \\ + (p-1)\frac{g''(\psi_*)}{g(\psi_*)}|\nabla'\psi_*|^2.\end{aligned} \tag{3.35}$$

In this expression the difficult term to deal with is $[(p-4)\beta_*\Lambda_{\beta_*}\psi_* - 2\Delta'\psi_*]$ since it has not a prescribed sign. However $\Delta'\psi_* = O(\psi_*)$ by (6.19) in Appendix II.

Step 2: The perturbation method and the computation with $g(\psi_*) = \psi_*$. With such a choice of function g

$$\begin{aligned}\mathcal{Q}_1[V] = (\gamma - \Lambda_{\beta_*})(\gamma - \beta_*) \left[1 + (p-2)\frac{\beta_*^2\psi_*^2}{\beta_*^2\psi_*^2 + |\nabla'\psi_*|^2}\right] - (p-1)\beta_*\Lambda_{\beta_*} \\ - (p-2)\left[(\gamma - \Lambda_{\beta_*})\beta_* + 2\gamma\right]\frac{|\nabla'\psi_*|^2}{\beta_*^2\psi_*^2 + |\nabla'\psi_*|^2} + \gamma\, O(\psi_*^2).\end{aligned} \tag{3.36}$$

Equivalently

$$\begin{aligned}\mathcal{Q}_1[V] = \left[1 + (p-2)\frac{\beta_*^2\psi_*^2}{\beta_*^2\psi_*^2 + |\nabla'\psi_*|^2}\right]\left(\gamma^2 - (\Lambda_{\beta_*} + \beta_*)\gamma\right) \\ - \gamma\left[(p-2)(\beta_*+2)\frac{|\nabla'\psi_*|^2}{\beta_*^2\psi_*^2 + |\nabla'\psi_*|^2} + O(\psi_*^2)\right]\end{aligned}$$

and finally

$$\mathcal{Q}_1[V] = \left[1 + (p-2)\frac{\beta_*^2\psi_*^2}{\beta_*^2\psi_*^2 + |\nabla'\psi_*|^2}\right]\gamma\left[\gamma - (\Lambda_{\beta_*} + \beta_* + (p-2)(\beta_*+2)) + O(\psi_*^2)\right]. \tag{3.37}$$

Using the fact that $\beta_* > \frac{N-1}{p-1}$ if $1 < p < 2$ and $1 < \beta_* < \frac{N-1}{p-1}$ if $2 < p < N$ (see Theorem 6.1 in Appendix II), we have

$$\Lambda_{\beta_*} + \beta_* + (p-2)(\beta_*+2) \geq \begin{cases} \Lambda_{\beta_*} + \beta_*(p-1) & \text{if } p \geq 2 \\ N + 3(p-2) > N - 3 & \text{if } 1 < p < 2. \end{cases} \tag{3.38}$$

When $N = 2$, we have explicitly $\beta_* = \frac{1+2\sqrt{p^2-3p+3}}{3(p-1)}$ (see [9, Th 3.3]). Therefore for all $N \geq 2$ and $p > 1$, there holds

$$\Lambda_{\beta_*} + \beta_* + (p-2)(\beta_*+2) > 0. \tag{3.39}$$

We fix $\epsilon_0 > 0$ such that, whenever $\psi_* \leq \epsilon_0$, there holds

$$\Lambda_{\beta_*} + \beta_* + (p-2)(\beta_*+2) + O(\psi_*^2) > \frac{1}{2}\left(\Lambda_{\beta_*} + \beta_* + (p-2)(\beta_*+2)\right). \tag{3.40}$$

If we fix $\gamma_0 > 0$ such that

$$\gamma_0 < \min\left\{\frac{1}{2}\left(\Lambda_{\beta_*} + \beta_* + (p-2)(\beta_* + 2)\right), \nu, \beta_*\right\}, \tag{3.41}$$

we obtain

$$\mathcal{Q}_1[V] \leq -\min\{1, p-1\}\gamma m^2 \qquad \forall\, 0 < \gamma \leq \gamma_0, \tag{3.42}$$

whenever $\psi_* \leq \epsilon_0$, for some m depending only on p, q and N (through ψ_* and ν), which, in the same range of value of ψ_*, yields to

$$\left(\beta_*^2\psi_*^2 + |\nabla'\psi_*|^2\right)^{\frac{p-2}{2}} g(\psi_*)\mathcal{Q}_1[V] \leq -c_{17}\psi_* \qquad \forall\, 0 < \gamma \leq \gamma_0, \tag{3.43}$$

for some $c_{17} > 0$ depending on N, p, q. This estimate is valid whatever is $p > 1$, but only in a neighborhood of $\psi_* = 0$. If we replace $g(\psi_*) = \psi_*$ by $g_k(\psi_*) = \psi_* e^{-k\psi_*}$ for $0 < k < 1$, and denote by $\mathcal{Q}_{1,k}[V]$ the corresponding expression of $\mathcal{Q}_1[V]$ which becomes now $\mathcal{Q}_{1,0}[V]$. We define similarly $\mathcal{Q}_k[V]$, and $\mathcal{Q}[V]$ becomes $\mathcal{Q}_0[V]$. Since $g_k'(\psi_*) = e^{-k\psi_*} - kg_k(\psi_*)$ and $g_k'' = -2ke^{-k\psi_*} + k^2 g_k(\psi_*)$, we obtain

$$\begin{aligned}\mathcal{Q}_{1,k}[V] = \mathcal{Q}_{1,0}[V] + k(p-1)\beta_*\Lambda_{\beta_*}\psi_* + (p-1)\left(-\frac{2k}{\psi_*} + k^2\right)|\nabla'\psi_*|^2 \\ + (2-p)\beta_*\left(-2k + k^2\right)\psi_* + O(\psi_*^2)\end{aligned} \tag{3.44}$$

Notice that $\nabla'\psi_*$ vanishes only at the North pole $\mathbf{e}_N$, thus there exists $k_0 \in (0, 1]$ such that

$$k(1-p)\beta_*\Lambda_{\beta_*}\psi_* + (p-1)\left(\frac{2k}{\psi_*} - k^2\right)\left|\nabla'\psi_*\right|^2 \geq \frac{1}{2}(2-p)_+\beta_*\left(-2k + k^2\right)\psi_* \qquad \forall k \leq k_0$$

whenever $\psi_* \leq \epsilon_0$ which yields to

$$\left(\beta_*^2\psi_*^2 + |\nabla'\psi_*|^2\right)^{\frac{p-2}{2}} g_k(\psi_*)\mathcal{Q}_{1,k}[V] \leq -c_{18}k \qquad \forall k \leq k_0 \tag{3.45}$$

for some $c_{13} = c_{13}(N, p, q, \epsilon_0)$. There exists $c_{14} = c_{14}(N, p, q) > 0$ such that

$$\left(\beta_*^2\psi_*^2 + |\nabla'\psi_*|^2\right)^{\frac{q}{2}}\left[1 + qe^{\gamma t}\frac{\beta_*(\gamma - \beta_*)\psi_* g_k(\psi_*) - g_k'(\psi_*)\left|\nabla\psi_*\right|^2}{\beta_*^2\psi_*^2 + |\nabla'\psi_*|^2}\right] \leq c_{14} \tag{3.46}$$

in $S_+^{N-1} \times (-\infty, \ln\delta]$. Moreover

$$O(a\,\|g(\psi_*)\|_{C^2}) \leq e^{\gamma t}\tilde{c}_k \tag{3.47}$$

for some $\tilde{c}_k = \tilde{c}_k(N, p, q) > 0$. We derive from (3.45)-(3.47)

$$e^{-\gamma t}\mathcal{Q}_k[V] \leq -c_{13}k + c_{14}e^{(\nu-\gamma)t} + e^{\gamma t}\tilde{c}_k \qquad \forall k \leq k_0 \tag{3.48}$$

Thus there exists $T_k \leq \ln\delta$ such that $\mathcal{Q}_k[V] \leq 0$, for all $t \leq T_k$ and provided $\psi_* \leq \epsilon_0$. This local estimate will be used in the construction of the subsolution when $p \geq 2$.

Step 3: The case $1 < p < 2$. Since the function ψ^* depends only on the azimuthal angle $\theta \in (0; \frac{\pi}{2}]$ we will write $\psi_*(\sigma) = \psi_*(\theta)$ and $\nabla'\psi_*(\sigma) = \psi_{*\theta}(\theta)\mathbf{n}$ where $\mathbf{n}$ is the downward unit vector tangent to S^{N-1} in the hyperplane going through σ and the poles. From (6.8),

$$(p-4)\beta_*\Lambda_{\beta_*}\psi_* - 2\Delta'\psi_* = (p-2)\left(\beta_*\Lambda_{\beta_*}\psi_* + 2\frac{\beta_*^2\psi_* + \psi_{*\theta\theta}}{\beta_*^2\psi_*^2 + \psi_{*\theta}^2}\right), \tag{3.49}$$

since $\psi_{*\theta}^2 = |\nabla'\psi_*|^2$ and thus

$$\begin{aligned}&\left((p-4)\beta_*\Lambda_{\beta_*}\psi_* - 2\Delta'\psi_*\right)\frac{\beta_*\gamma\psi_*}{\beta_*^2\psi_*^2 + \psi_{*\theta}^2}\\ &\qquad = (p-2)\gamma\left(\Lambda_{\beta_*}\frac{\beta_*^2\psi_*^2}{\beta_*^2\psi_*^2 + \psi_{*\theta}^2} + 2\beta_*\frac{\beta_*^2\psi_*^2 + \psi_{*\theta\theta}\psi_*}{(\beta_*^2\psi_*^2 + \psi_{*\theta}^2)^2}\right).\end{aligned} \tag{3.50}$$

From Theorem 6.1-Step 4 in Appendix II, we know that $\beta_*^2\psi_* + \psi_{*\theta\theta} \geq 0$, thus the contribution of this term to $\mathcal{Q}_1[V]$ is nonpositive. We replace this expression in $\mathcal{Q}_1[V]$ with $g(\psi_*) = \psi_*$ and obtain

$$\begin{aligned}\mathcal{Q}_1[V] &= (\gamma - \Lambda_{\beta_*})(\gamma - \beta_*)\left(1 + (p-2)\frac{\beta_*^2\psi_*^2}{\beta_*^2\psi_*^2 + \psi_{*\theta}^2}\right) - \Lambda_{\beta_*}\beta_*(p-1)\\ &\quad + (p-2)\gamma\Lambda_{\beta_*}\frac{\beta_*^2\psi_*^2}{\beta_*^2\psi_*^2 + \psi_{*\theta}^2} - (p-2)\left((\beta_* + 2)\gamma - \Lambda_{\beta_*}\beta_*\right)\frac{\psi_{*\theta}^2}{\beta_*^2\psi_*^2 + \psi_{*\theta}^2}\\ &\quad + 2\beta_*(p-2)\frac{\beta_*^2\psi_*^2 + \psi_{*\theta\theta}\psi_*}{(\beta_*^2\psi_*^2 + \psi_{*\theta}^2)^2}\gamma\\ &\leq \gamma\left(1 + (p-2)\frac{\beta_*^2\psi_*^2}{\beta_*^2\psi_*^2 + \psi_{*\theta}^2}\right)(\gamma - \Lambda_{\beta_*} - \beta_*) - (p-2)\gamma\frac{(\beta_* + 2))\psi_{*\theta}^2 - \Lambda_{\beta_*}\beta_*^2\psi_*^2}{\beta_*^2\psi_*^2 + \psi_{*\theta}^2}\\ &\leq \gamma\left(1 + (p-2)\frac{\beta_*^2\psi_*^2}{\beta_*^2\psi_*^2 + \psi_{*\theta}^2}\right)\left(\gamma - \left(\Lambda_{\beta_*} + \beta_* + (p-2)\frac{(\beta_* + 2)\psi_{*\theta}^2 - \Lambda_{\beta_*}\beta_*^2\psi_*^2}{(p-1)\beta_*^2\psi_*^2 + \psi_{*\theta}^2}\right)\right).\end{aligned} \tag{3.51}$$

We can write

$$\begin{aligned}&\Lambda_{\beta_*} + \beta_* + (p-2)\frac{(\beta_* + 2)\psi_{*\theta}^2 - \Lambda_{\beta_*}\beta_*^2\psi_*^2}{(p-1)\beta_*^2\psi_*^2 + \psi_{*\theta}^2}\\ &\qquad = \frac{\left(\Lambda_{\beta_*} + (p-1)\beta_*\right)\beta_*^2\psi_*^2 + \left(\Lambda_{\beta_*} + \beta_*(p-1) + 2(p-2)\right)\psi_{*\theta}^2}{(p-1)\beta_*^2\psi_*^2 + \psi_{*\theta}^2}\\ &\qquad \geq c_{15}\left(\Lambda_{\beta_*} + \beta_*(p-1) + 2(p-2)\right)\end{aligned} \tag{3.52}$$

for some positive constant c_{15}. This expression $\Lambda_{\beta_*} + \beta_*(p-1) + 2(p-2)$ is always positive: obviously if $N \geq 3$ and by using the explicit expression of β_* if $N = 2$. Thus there exists γ_0 and $c_{16} > 0$ such that $\mathcal{Q}_1[V] < -c_{16}$ for $0 < \gamma \leq \gamma_0$. The perturbation method of Step 2, is valid in the whole range of values of ψ_* and we derive from (3.42)-(3.43) that (3.48) holds for all $k \leq k_0$ and $t \leq T_k$. Therefore $\mathcal{Q}_k[V] \leq 0$.

Step 4: The case $p \geq 2$. For $c > 0$ to be fixed and $\psi_* \geq \epsilon_0$, $\gamma \in (0, \gamma_0]$, we take $g(\psi_*) = c\psi_*^{1-\frac{\gamma}{\beta_*}}$. Then we derive from (3.35):

$$\begin{aligned}\mathcal{Q}_1[V] &= (\gamma-\Lambda_{\beta_*})(\gamma-\beta_*)\frac{(p-1)\beta_*^2\psi_*^2+|\nabla'\psi_*|^2}{\beta_*^2\psi_*^2+|\nabla'\psi_*|^2}-(p-1)\beta_*\Lambda_{\beta_*}\left(1-\frac{\gamma}{\beta_*}\right)\\ &\quad-(p-1)\frac{\gamma(\beta_*-\gamma)}{\beta_*^2}\psi_*^{-1-\frac{\gamma}{\beta_*}}\left|\nabla'\psi_*\right|^2-(p-2)(\beta_*-\gamma)(\gamma-\Lambda_{\beta_*})\frac{\left|\nabla'\psi_*\right|^2}{\beta_*^2\psi_*^2+|\nabla'\psi_*|^2}\\ &= (1-p)\left[\gamma(\beta_*-\gamma)+\frac{\gamma(\beta_*-\gamma)}{\beta_*^2}\psi_*^{-1-\frac{\gamma}{\beta_*}}\left|\nabla'\psi_*\right|^2\right].\end{aligned} \tag{3.53}$$

For $k\leq k_0$ we fix c such that $c\epsilon_0^{1-\frac{\gamma}{\beta_*}}=\epsilon_0e^{-k\epsilon_0}\Longleftrightarrow c=\epsilon_0^{\frac{\gamma}{\beta_*}}e^{-k\epsilon_0}$ and we define g by

$$g(\psi_*)=\min\left\{\psi_*e^{-k\psi_*},\epsilon_0^{\frac{\gamma}{\beta_*}}e^{-k\epsilon_0}\psi_*^{1-\frac{\gamma}{\beta_*}}\right\}=\left\{\begin{array}{ll}\psi_*e^{-k\psi_*} & \text{if } 0\leq\psi_*\leq\epsilon_0\\ \epsilon_0^{\frac{\gamma}{\beta_*}}e^{-k\epsilon_0}\psi_*^{1-\frac{\gamma}{\beta_*}} & \text{if } \epsilon_0\leq\psi_*\leq 1,\end{array}\right. \tag{3.54}$$

and we set $V(t,\sigma)=\psi^*(\sigma)-a(t)g(\psi_*(\sigma))$ with $(t,\sigma)\in(-\infty,T_k]\times S_+^{N-1}$ and define $\tilde u(r,\sigma)=r^{-\beta_*}(\psi^*(\sigma)-a(\ln r)g(\psi_*(\sigma)))$ accordingly for $(r,\sigma)\in(-\infty,e^{T_k}]\times S_+^{N-1}$. Since ψ_* is a decreasing function the coincidence set $\{\sigma\in S_+^{N-1}:\psi_*(\sigma)=\epsilon_0\}$ is a circular cone Σ_{θ_0} with vertex 0, axis $\mathbf{e}_N$ and angle θ_0. We set $R_0=e^{T_k}$

$$\begin{aligned}\Gamma_1&=\left\{x=(r,\theta)\in B_{R_0}^+:\theta_0<\theta<\tfrac{\pi}{2}\right\}=\left\{(r,\sigma)\in[0,R_0)\times S_+^{N-1}:0<\psi_*(\sigma)<\epsilon_0\right\},\\ \Gamma_2&=\left\{x=(r,\theta)\in B_{R_0}^+:0<\theta<\theta_0\right\}=\left\{(r,\sigma)\in[0,R_0)\times S_+^{N-1}:\epsilon_0<\psi_*(\sigma)<1\right\},\end{aligned}$$

and define

$$\begin{aligned}\tilde u(r,\sigma)&=r^{-\beta_*}\left(\psi_*(\sigma)-r^\gamma g(\psi_*(\sigma))\right)\\ &=\left\{\begin{array}{ll}u_1(r,\sigma)=r^{-\beta_*}(1-r^\gamma e^{-k\psi_*(\sigma)})\psi_*(\sigma) & \text{if } (r,\theta)\in\Gamma_1\\ u_2(r,\sigma)=r^{-\beta_*}\left(1-r^\gamma\epsilon_0^{\frac{\gamma}{\beta_*}}e^{-k\epsilon_0}(\psi_*(\sigma))^{1-\frac{\gamma}{\beta_*}}\right)\psi_*(\sigma) & \text{if } (r,\theta)\in\Gamma_2.\end{array}\right.\end{aligned}$$

The function $\tilde u$ is a subsolution separately on Γ_1 and Γ_2 and is Lipschitz continuous in $\overline{\Omega}\setminus\{0\}$. If we denote by g_1 and g_2 the restriction of g to Γ_1 and Γ_2 respectively, that is to Ω_1 and Ω_2, then $g_1'(\epsilon_0)>g_2'(\epsilon_0)>0$. Let $\zeta\in C_c^1(B_{R_0}^+)$ which vanishes in neighborhoods of 0 and $\partial B_{R_0}^+$, $\zeta\geq 0$, then

$$\int_{\Gamma_i}|\nabla\tilde u|^{p-2}\nabla\tilde u.\nabla\zeta dx+\int_{\Omega_i}|\nabla\tilde u|^q\zeta dx\leq\int_{\Sigma_{\theta_0}}|\nabla u_i|^{p-2}\partial_{\mathbf{n}_i}u_i\zeta dS, \tag{3.55}$$

where $\mathbf{n}_i$ is the normal unit vector on Σ_{θ_0} outward from Γ_i. Actually, $\mathbf{n}_2=-\mathbf{n}_1=\mathbf{n}$ thus

$$\nabla\tilde u=\tilde u_r\mathbf{e}+r^{-\beta_*-1}(1-r^\gamma g'(\psi_*))\nabla'\psi_*=\tilde u_r\mathbf{e}+r^{-\beta_*-1}(1-r^\gamma g'(\psi_*))\psi_{*\theta}\,\mathbf{n}.$$

and on Σ_{θ_0},

$$\nabla\tilde u=\left\{\begin{array}{ll}\tilde u_r\mathbf{e}-r^{-\beta_*-1}(1-r^\gamma g_1'(\epsilon_0))\psi_{*\theta}\,\mathbf{n} & \text{in } \Gamma_1\\ \tilde u_r\mathbf{e}+r^{-\beta_*-1}(1-r^\gamma g_2'(\epsilon_0))\psi_{*\theta}\,\mathbf{n} & \text{in } \Gamma_2\end{array}\right.$$

Therefore

$$\begin{aligned}&|\nabla u_1|^{p-2}\,\partial_{\mathbf{n}_1}u_1\\&\qquad= -r^{-\beta_*-1}(1-r^\gamma g_1'(\epsilon_0))\left(\tilde u_r^2+r^{-2\beta_*-2}(1-r^\gamma g_1'(\epsilon_0))^2\psi_{*\theta}^2\right)^{\frac{p-2}{2}}\psi_{*\theta}\quad\text{ in }\Gamma_1\end{aligned}$$

and

$$\begin{aligned}&|\nabla u_2|^{p-2}\,\partial_{\mathbf{n}_2}u_2\\&\qquad= r^{-\beta_*-1}(1-r^\gamma g_2'(\epsilon_0))\left(\tilde u_r^2+r^{-2\beta_*-2}(1-r^\gamma g_2'(\epsilon_0))^2\psi_{*\theta}^2\right)^{\frac{p-2}{2}}\psi_{*\theta}\quad\text{ in }\Gamma_2.\end{aligned}$$

By adding the two inequalities (3.55)

$$\int_\Omega|\nabla\tilde u|^{p-2}\nabla\tilde u.\nabla\zeta dx+\int_\Omega|\nabla\tilde u|^q\zeta dx\leq\int_{\Sigma_{\theta_0}}\left(|\nabla u_1|^{p-2}\,\partial_{\mathbf{n_1}}u_1+|\nabla u_2|^{p-2}\,\partial_{\mathbf{n_2}}u_2\right)\zeta dS.\tag{3.56}$$

By monotonicity of the function $X\mapsto\left(\tilde u_r^2+X^2\right)^{\frac p2}$ and since

$$r^{-\beta_*-1}(1-r^\gamma g_2'(\epsilon_0))\geq r^{-\beta_*-1}(1-r^\gamma g_1'(\epsilon_0))\geq 0,$$

we derive

$$\begin{aligned}&r^{-\beta_*-1}(1-r^\gamma g_2'(\epsilon_0))\left(\tilde u_r^2+r^{-2\beta_*-2}(1-r^\gamma g_2'(\epsilon_0))^2\psi_{*\theta}^2\right)^{\frac{p-2}{2}}\\&\qquad\geq r^{-\beta_*-1}(1-r^\gamma g_1'(\epsilon_0))\left(\tilde u_r^2+r^{-2\beta_*-2}(1-r^\gamma g_1'(\epsilon_0))^2\psi_{*\theta}^2\right)^{\frac{p-2}{2}}\end{aligned}$$

We derive that the right-hand side of (3.56) is nonpositive because $\psi_{*\theta}\leq 0$, and therefore $\tilde u$ is a positive subsolution of (1.1) in $B^+_{R_0}$ dominated by Ψ_* and satisfying (3.24). □

Proof of Theorem 3.6. Let $M=\max\{\Psi_*(x):x\in\partial B^+_{R_0}\}$, then $M=R_0^{-\beta_*}$. The function u^* defined by

$$u^*(x)=\begin{cases}(\tilde u(x)-M)_+ & \text{if } x\in B^+_{R_0}\\ 0 & \text{if } x\in\Omega\setminus B^+_{R_0},\end{cases}$$

is indeed a subsolution of (1.1) in whole Ω where it satisfies $u^*\leq\Psi_*$ and it vanishes on $\partial\Omega\setminus\{0\}$. Since Φ_* is a positive p-harmonic function in Ω which vanishes on $\partial\Omega\setminus\{0\}$ and satisfies (3.20), it is supersolution of (1.1) and therefore it dominates u^*. Therefore there exists a solution u of (1.1) in Ω which vanishes on $\partial\Omega\setminus\{0\}$ and satisfies $u^*\leq u\leq\Phi_*$. This implies that (3.19) holds with $k=1$ and we conclude with Lemma 3.8. This ends the proof of Lemma 3.9. □

When $p=N$ the statement of Theorem 3.6 holds without the flatness assumption on $\partial\Omega$. The proof of the next theorem is an easy adaptation to the one of Theorem 3.6, provided Lemma 3.7, Lemma 3.8 and Lemma 3.9 are modified accordingly.

Theorem 3.10. *Assume $N-1<q<N-\frac12$ and Ω be a bounded C^2 domain such that $0\in\partial\Omega$. Then for any $k>0$ there exists a unique positive solution $u:=u_k$ of* (3.17) *in Ω, which belongs to $C^1(\overline\Omega\setminus\{0\})$, vanishes on $\partial\Omega\setminus\{0\}$ and satisfies uniformly with respect to $\sigma\in S^{N-1}_+$*

$$\lim_{\substack{x\to0\\ x/|x|\to\sigma}}|x|\,u_k(x)=k\psi_*(\sigma).\tag{3.57}$$

Since $p = N$, then $\beta_* = 1$ and $\psi_*(\sigma) = \frac{x_N}{|x|} = \cos\theta_{N-1}$ with the identification of σ and $\theta_{N-1} := \theta$. In a more intrinsic manner (3.57) can be written under the form

$$\lim_{\substack{x \to 0 \\ x \in \Omega}} |x|^2 \frac{u_k(x)}{d(x)} = k. \tag{3.58}$$

We recall that if $\omega \in \mathbb{R}^N$ and $\mathcal{I}_\omega$ denotes the inversion of center ω and power 1, i.e. $\mathcal{I}_\omega(x) = \omega + \frac{x-\omega}{|x-\omega|^2}$, then $\tilde{u} = u \circ \mathcal{I}_\omega$ satisfies (3.18).

Lemma 3.11. *Assume Ω be a bounded C^2 domain such that $0 \in \partial\Omega$. Then there exists a unique N-harmonic function Φ_* in Ω, which vanishes on $\partial\Omega \setminus \{0\}$ and satisfies*

$$\lim_{\substack{x \to 0 \\ x/|x| \to \sigma}} |x|\,\Phi_*(x) = \psi_*(\sigma), \tag{3.59}$$

uniformly with respect to $\sigma \in S_+^{N-1}$.

Proof. Uniqueness is standard. Let $\omega = -\mathbf{e}_N \in \overline{\Omega}^c$, with the notations of the proof of Theorem 3.5, $\omega' = -\omega$, $a = -\frac{1}{2}\mathbf{e}_N$ and $a' = -a$. We can assume that the balls $B_{\frac{1}{2}}(a)$ and $B_{\frac{1}{2}}(a')$ are tangent to $\partial\Omega$ at 0 and respectively subset of Ω^c and Ω. The function $x \mapsto \Psi(x) = -\frac{x_N}{|x|^2}$ which is N-harmonic in $\mathbb{R}_-^N$ and vanishes on $\partial\mathbb{R}_-^N \setminus \{0\}$ is transformed by the inversion $\mathcal{I}_{\omega'}$ of center ω' and power 1 into the function $\Psi_{\omega'} = \Psi \circ \mathcal{I}_\omega$ which is positive and N-harmonic in $B_{\frac{1}{2}}(a')$ and vanishes on $\partial B_{\frac{1}{2}}(a') \setminus \{0\}$. The function $\hat{\Psi} = -\Psi$ which is N-harmonic in $\mathbb{R}_+^N$ and vanishes on $\partial\mathbb{R}_+^N \setminus \{0\}$ is transformed by the inversion $\mathcal{I}_\omega$ of center ω and power 1 into the function $\Psi_\omega = \hat{\Psi} \circ \mathcal{I}_\omega$ which is positive and N-harmonic in $B^c_{\frac{1}{2}}(a)$ and vanishes on $\partial B_{\frac{1}{2}}(a) \setminus \{0\}$. For $\epsilon > 0$ we denote by Φ_ϵ the solution of

$$\begin{aligned} -\Delta_N \Phi_\epsilon &= 0 && \text{in } \Omega \cap B_\epsilon^c \\ \Phi_\epsilon &= 0 && \text{in } (B^c_{\frac{1}{2}}(a') \cap \partial B_\epsilon) \cup (\partial\Omega \cap B_\epsilon^c) \\ \Phi_\epsilon &= \Psi_{\omega'} && \text{in } B_{\frac{1}{2}}(a') \cap \partial B_\epsilon. \end{aligned} \tag{3.60}$$

If $0 < \epsilon' < \epsilon$, $\Phi_{\epsilon'} \geq \Psi_{\omega'}$ in $B_{\frac{1}{2}}(a') \cap \partial B_\epsilon$, thus $\Phi_{\epsilon'} \geq \Phi_{\epsilon'}$ in $\Omega \cap B_\epsilon^c$. We also denote by $\hat{U}_\epsilon$ the solution of

$$\begin{aligned} -\Delta_N \hat{\Phi}_\epsilon &= 0 && \text{in } \Omega \cap B_\epsilon^c \\ \hat{\Phi}_\epsilon &= 0 && \text{in } \partial\Omega \cap B_\epsilon^c \\ \hat{\Phi}_\epsilon &= \Psi_\omega && \text{in } \Omega \cap \partial B_\epsilon^c. \end{aligned} \tag{3.61}$$

In the same way as above

$$0 < \epsilon' < \epsilon \Longrightarrow \hat{\Phi}_{\epsilon'} \leq \hat{\Phi}_\epsilon \quad \text{in } \Omega \cap \partial B_\epsilon^c$$

Using the explicit form of $\Psi, \mathcal{I}_\omega : x \mapsto \omega + \frac{x-\omega}{|x-\omega|^2}$ and $\mathcal{I}_{\omega'} : x \mapsto \omega' + \frac{x-\omega'}{|x-\omega'|^2}$ we see that

$$\Psi_{\omega'} \lfloor_{B_{\frac{1}{2}}(a') \cap \partial B_\epsilon} \leq \frac{1+\epsilon}{1-\epsilon} \Psi_\omega \lfloor_{B_{\frac{1}{2}}(a') \cap \partial B_\epsilon},$$

thus

$$\Phi_\epsilon \leq \frac{1+\epsilon}{1-\epsilon} \hat{\Phi}_\epsilon \quad \text{in } \Omega \cap B_\epsilon^c.$$

Letting $\epsilon \to 0$ we conclude that Φ_ϵ converges uniformly in $\overline{\Omega} \setminus \{0\}$ to Φ_* which vanishes on $\partial\Omega \setminus \{0\}$ and satisfies (3.59). $\square$

The proof of the next statement is similar to the one of Lemma 3.8 up to some minor modifications, so we omit it.

Lemma 3.12. *Let the assumptions on q and Ω of Theorem 3.10 be satisfied. If for some $k > 0$ there exists a solution u_k of* (3.17) *in Ω, which belongs to $C^1(\overline{\Omega} \setminus \{0\})$, vanishes on $\partial\Omega \setminus \{0\}$ and satisfies* (3.57)*, then for any $k > 0$ there exists such a solution.*

Lemma 3.13. *Under the assumptions of Theorem 3.10 there exists a Lipschitz continuous nonnegative subsolution $\tilde{u}$ of* (3.17) *in Ω which vanishes on $\partial\Omega \setminus \{0\}$, is smaller than Φ_* and satisfies*

$$\lim_{\substack{x \to 0 \\ x/|x| \to \sigma}} |x|\, \tilde{u}(x) = \sigma, \tag{3.62}$$

uniformly with respect to $\sigma \in S_+^{N-1}$.

Proof. Let $\tau > 0$ to be fixed and let w be the solution of

$$-\Delta_N w + |\nabla w|^q = 0 \qquad \text{in } B_2^- \tag{3.63}$$

which vanishes on $\partial B_2^- \setminus \{0\}$ and satisfies

$$\lim_{\substack{x \to 0 \\ x/|x| \to \sigma}} |x| w(x) = \sigma \tag{3.64}$$

in the C^1-topology of S_-^{N-1}. Its existence follows from Theorem 3.6 and this function is dominated by the N-harmonic function Φ_* corresponding to this domain, obtained in Lemma 3.11. By $\mathcal{I}_{\omega'}$, the half-ball B_2^- is transform into the lunule $G = B_{\frac{1}{2}}(a') \setminus B_{\frac{2}{3}}(\frac{4}{3}\omega')$ and $\tilde{w} = w \circ \mathcal{I}_{\omega'}$ satisfies

$$-\Delta_N \tilde{w} + |x - \omega'|^{2(q-N)} |\nabla \tilde{w}|^q = 0 \qquad \text{in } G. \tag{3.65}$$

Since $|x - \omega'| \leq 1$ in G, $-\Delta_N \tilde{w} + |\nabla \tilde{w}|^q \leq 0$ in G. We extend $\tilde{w}$ by 0 in $\Omega \setminus G$ and the resulting function $\tilde{u}$ is a subsolution of (3.17) in Ω which vanishes on $\partial\Omega \setminus \{0\}$), is smaller than the N-harmonic function Φ_* obtained in Lemma 3.11, and satisfies (3.62). $\square$

4 Classification of boundary singularities

We assume that $\Omega \subset \mathbb{R}^N$ is a C^2 domain and $0 \in \partial\Omega$. Furthermore, in order to avoid extremely technical computations, we shall assume either that $\partial\Omega$ is flat near 0 or $p = N$. We suppose that the tangent plane to $\partial\Omega$ at 0 is $\partial\mathbb{R}_+^N = \{x = (x', 0)\}$ and the normal inward unit vector at 0 is $\mathbf{e}_N$, therefore $\mathbf{n} = -\mathbf{e}_N$ in the sequel. We denote by $\omega_{s_+^{N-1}}$ the unique positive solution of (3.1) in S_+^{N-1} and by $U_{s_+^{N-1}}$ the corresponding singular solution of (1.1) in $\mathbb{R}_+^N$ defined by

$$U_{s_+^{N-1}}(x) = |x|^{-\beta_q}\, \omega_{s_+^{N-1}}(\frac{x}{|x|}). \tag{4.66}$$

We recall that ψ_* is the unique positive solution of (3.2) with maximum 1 and Ψ_* the corresponding p-harmonic function

$$\Psi_*(x) = |x|^{-\beta_*}\,\psi_*(\frac{x}{|x|}). \tag{4.67}$$

4.1 The case $1 < p < N$

The first statement points out the link between weak and strong singularities.

Proposition 4.1. *Under the assumptions of Theorem 3.6 there exists* $\lim_{k\to\infty} u_k = u_\infty$ *which is the unique element of* $C(\overline{\Omega}\setminus\{0\})\cap C^1(\Omega)$ *which vanishes on* $\partial\Omega\setminus\{0\}$*, satisfies* (1.1) *in* Ω *and*

$$\lim_{x\to 0}\frac{u_\infty(x)}{U_{s_+^{N-1}}(x)} = 1. \tag{4.68}$$

Proof. Uniqueness follows from (4.68) and the maximum principle. For existence, since the mapping $k\mapsto u_k$ is increasing and $u_k\leq U_{s_+^{N-1}}$, there exists $\lim_{k\to\infty} u_k := u_\infty \leq U_{s_+^{N-1}}$ and $u_\infty\in C(\overline{\Omega}\setminus\{0\})\cap C^1(\Omega)$. It vanishes on $\partial B_\delta^+\setminus\{0\}$ and satisfies (1.1) in B_δ^+. In order to take into account the domain B_δ^+ in the notations, we set $u_k = u_{k,\delta}$. Since the mapping $\delta\mapsto u_{k,\delta}$ is also increasing and $u_{k,\delta}\leq k\Psi_*$, there also exists $\lim_{\delta\to\infty} u_{k,\delta} := u_{k,\infty}\leq k\Psi_*$ Then, for all $\ell > 0$,

$$T_\ell[u_{k,\delta}](x) = \ell^{\beta_q} u_{k,\delta}(\ell x) = u_{k\ell^{\beta_q},\ell^{-1}\delta}(x). \tag{4.69}$$

Letting $k\to\infty$, we obtain

$$T_\ell[u_{\infty,\delta}](x) = \ell^{\beta_q} u_{\infty,\delta}(\ell x) = u_{\infty,\ell^{-1}\delta}(x), \tag{4.70}$$

and letting $\delta\to\infty$, we obtain

$$T_\ell[u_{\infty,\infty}](x) = \ell^{\beta_q} u_{\infty,\infty}(\ell x) = u_{\infty,\infty}(x). \tag{4.71}$$

This implies that

$$u_{\infty,\infty}(r,\sigma) = r^{-\beta_q}\omega'(\sigma), \tag{4.72}$$

and ω' is a positive solution of problem (3.1). Therefore $\omega' = \omega_{s_+^{N-1}}$ by Theorem 3.2. If we let $\ell\to 0$ in (4.69) and take $|x| = 1$, $x = \sigma$, we derive

$$\lim_{\ell\to 0}\ell^{\beta_q} u_{\infty,\delta}(\ell,\sigma) = \lim_{\ell\to 0} u_{\infty,\ell^{-1}\delta}(1,\sigma) = u_{\infty,\infty}(1,\sigma) = \omega_{s_+^{N-1}}(\sigma). \tag{4.73}$$

This convergence holds in $C^1(\overline{S_+^{N-1}})$ because of Lemma 2.5. This implies (4.68). $\square$

The main classification result is as follows.

Theorem 4.2. *Assume* $1 < p < N$, $p-1 < q < q^*$ *and* $\partial\Omega\cap B_\delta = \{x = (x',0) : |x'| < \delta\}$*, for some* $\delta > 0$*. If* $u\in C(\overline{\Omega}\setminus\{0\})\cap C^1(\Omega)$ *is a positive solution of* (1.1) *in* Ω *which vanishes on* $\partial\Omega\setminus\{0\}$*, then we have the following alternative:*

(i) either there exists $k\geq 0$ *such that*

$$\lim_{x\to 0}\frac{u(x)}{\Psi_*(x)} = k, \tag{4.74}$$

(ii) or

$$\lim_{x\to 0}\frac{u(x)}{U_{s_+^{N-1}}(x)}=1. \tag{4.75}$$

Proof. Step 1. Assume

$$\liminf_{x\to 0}\frac{u(x)}{\Psi_*(x)}<\infty, \tag{4.76}$$

then we claim that (4.74) holds. We first note that if (4.76) holds, there also holds

$$\liminf_{x\to 0}\frac{u(x)}{u_1(x)}<\infty, \tag{4.77}$$

where u_1 is the solution of (1.1) obtained in Theorem 3.6 with $k=1$. If $\{x_n\}$ is converging to 0 and such that for some $k>0$

$$\liminf_{x\to 0}\frac{u(x)}{u_1(x)}=k=\lim_{n\to\infty}\frac{u(x_n)}{u_1(x_n)},$$

there also holds by the boundary Harnack inequality (2.38) applied to both u and u_1,

$$\frac{u(x_n)}{u_1(x_n)}=\frac{u(x_n)}{d(x_n)}\frac{d(x_n)}{u_1(x_n)}\geq c_5^{-2}\frac{u(x)}{u_1(x)}\quad \forall x \text{ s.t. } |x|=|x_n|\,.$$

This implies in particular

$$u(x)\leq c_5^2(k+\epsilon_n)u_1(x)\qquad \forall x \text{ s.t. } |x|=|x_n|$$

where $\{\epsilon_n\}$ is converging to 0_+, and by the comparison principle

$$u(x)\leq Ku_1(x)\qquad \forall x\in\mathbb{R}_+^N \text{ s.t. } |x_n|\leq|x|\leq\frac{\delta}{2},$$

for some $K>0$ and all $n\in\mathbb{N}_*$. Therefore

$$\limsup_{x\to 0}\frac{u(x)}{u_1(x)}<\infty. \tag{4.78}$$

We can assume that $k\neq 0$, otherwise (4.74) holds with $k=0$ and actually u remains bounded near 0. As a consequence of the Hopf Lemma and C^1 regularity, there exists $K>0$ such that

$$u(x)\leq K\Psi_*(x)\qquad \forall x\in B^+_{\frac{\delta}{2}}. \tag{4.79}$$

Let $m=\max\{u(x):|x|=\delta\}$. For $0<\tau<\delta$ we denote by k_τ the minimum of the $\kappa>0$ such that $u(x)\leq\kappa\Psi_*(x)+m$ for $\tau\leq|x|\leq\delta$. Then $u(x)\leq k_\tau\Psi_*(x)+m$, and either the graphs of the mappings $u(.)$ and $k_\tau\Psi_*(.)+m$ are tangent at some $x_\tau\in B_\delta^+\setminus\overline{B}_\tau^+$, or they are tangent on the boundary of the domain, and the only possibility is that they are tangent on $|x|=\tau$. Since

$$|\nabla\Psi_*(x)|^2=|x|^{-2(\beta_*+1)}\left(\beta_*^2\psi_*^2+|\nabla\psi_*|^2\right),$$

it never vanishes. If we set $w = u - (k_\tau \Psi_*(x) + m)$, then

$$-\mathcal{L}w + |\nabla u|^q = 0 \tag{4.80}$$

where the operator

$$\mathcal{L} = \sum_{i,j} \frac{\partial}{\partial x_i}\left(a_{ij}\frac{\partial}{\partial x_j}\right)$$

is uniformly elliptic in a neighborhood of x_τ (see [6, Lemma 1.3]). Furthermore $w \leq 0$ and $w(x_\tau) = 0$ by the strong maximum principle $\nabla u(x_\tau)$ must vanish, which contradicts the fact that $\nabla u(x_\tau) = \nabla w(x_\tau)$ by the tangency condition, and $\nabla w(x_\tau) \neq 0$. Therefore $|x_\tau| = \tau$ and $x_\tau \notin \partial\mathbb{R}^N_+$. If $\tau' < \tau$, $k_\tau \leq k_{\tau'}$, and we set $k = \lim_{\tau\to 0} k_\tau$, which is finite because of (4.79). There exists $\{\tau_n\}$ such that $\sigma_n := \tau^{-1}x_{\tau_n} \to \sigma_0$. Furthermore

$$r^{\beta_*}u(r,\sigma) \leq k_\tau\psi_*(\sigma) + mr^{\beta_*} \quad \text{if } \tau \leq r \leq \delta \quad \text{and } \tau^{\beta_*}u(\tau,\sigma_\tau) = k_\tau\psi_*(\sigma_\tau) + m\tau^{\beta_*}. \tag{4.81}$$

Put

$$u_\tau(x) = \tau^{\beta_*}u(\tau x) \tag{4.82}$$

Then

$$-\Delta_p u_\tau + \tau^{p-q-\beta_*(p+1-q)}\,|\nabla u_\tau|^q = 0 \qquad \text{in } B^+_{\frac{\delta}{\tau}} \setminus \{0\}$$

and, by (4.79),

$$0 \leq u_\tau(x) \leq K\,|x|^{-\beta_*} \qquad \text{in } B^+_{\frac{\delta}{2\tau}} \setminus \{0\}.$$

By Lemma 2.5, the set of functions $\{u_\tau(.)\}$ is relatively compact in the C^1_{loc} topology of $\overline{\mathbb{R}^N_+} \setminus \{0\}$. Therefore, as $q < q^*$, there exist a sequence $\{\tau'_n\} \subset \{\tau_n\}$ converging to 0, and a positive p-harmonic function v in $\mathbb{R}^N_+$, continuous in $\overline{\mathbb{R}^N_+}\setminus\{0\}$ and vanishing on $\partial\mathbb{R}^N_+\setminus\{0\}$, such that $u_{\tau'_n} \to v$, and v satisfies (4.79) in $\overline{\mathbb{R}^N_+} \setminus \{0\}$. By Theorem 5.1 in Appendix I, there exists k^* such that $v = k^*\Psi_*$. In particular,

$$\lim_{\tau'_n \to 0} u_{\tau'_n}(1,\sigma) = k^*\psi_*(\sigma) \tag{4.83}$$

in the $C^1(\overline{S^{N-1}_+})$ topology. Combining (4.81), (4.82)and (4.83) we conclude that $k^* = k$ and

$$\lim_{\tau'_n \to 0} \tau'^{\beta_*}_n u_{\tau'_n}(1,\sigma) = k\psi_*(\sigma) \tag{4.84}$$

Using Theorem 3.6, it is equivalent to

$$\lim_{\tau'_n \to 0} \frac{u(\tau'_n,\sigma)}{u_k(\tau'_n,\sigma)} = 1 \tag{4.85}$$

uniformly on S^{N-1}_+. For any $\epsilon > 0$, there exists $n_\epsilon > 0$ such that $n \geq n_\epsilon$ implies

$$u_{k-\epsilon}(\tau'_n,\sigma) \leq u(\tau'_n,\sigma) \leq u_{k+\epsilon}(\tau'_n,\sigma)$$

By comparison principle,

$$u_{k-\epsilon} \leq u \leq u_{k+\epsilon} + m \qquad \text{in } B^+_\delta \setminus B^+_{\tau'_n}, \tag{4.86}$$

and finally

$$u_{k-\epsilon} \leq u \leq u_{k+\epsilon} + m \qquad \text{in } B_\delta^+, \tag{4.87}$$

Since ϵ is arbitrary and using again Theorem 3.6, it implies

$$\lim_{r\to 0} \frac{u(r,\sigma)}{\Psi_*(r,\sigma)} = k, \tag{4.88}$$

locally uniformly on S^{N-1}. But since the convergence holds in $C^1(\overline{S_+^{N-1}})$, (4.74) follows.

Step 2. Assume

$$\lim_{x\to 0} \frac{u(x)}{\Psi_*(x)} = \infty. \tag{4.89}$$

For any $0 < \epsilon < \delta$ and $k > 0$, there holds

$$u_k(x) \leq u(x) \leq v_\epsilon(x) \qquad \text{in } B_\delta^+ \setminus B_\epsilon^+ \tag{4.90}$$

where v_ϵ has been defined in (3.12) and u_k is given by Theorem 3.6. Letting $\epsilon \to 0$, $k \to \infty$, and using Proposition 4.1, we derive

$$u_\infty(x) \leq u(x) \leq v_0(x) \qquad \text{in } B_\delta^+ \setminus \{0\}. \tag{4.91}$$

We have seen in Theorem 3.3 that v_0 is a separable solution of (1.1) in $\mathbb{R}_+^N$ which vanishes on $\partial\mathbb{R}_+^N \setminus \{0\}$, therefore $v_0(x) = U_{s_+^{N-1}}(x)$. This implies

$$u_\infty(x) \leq u(x) \leq |x|^{-\beta_q}\, \omega_{s_+^{N-1}}(\frac{x}{|x|}) \qquad \text{in } B_\delta^+ \setminus \{0\}. \tag{4.92}$$

We conclude using Proposition 4.1. □

4.2 The case $p = N$

When $p = N$, the assumption that $\partial\Omega$ is an hyperplane near 0 can be removed. The proof of the next results is based upon Theorem 3.10. The following result is the extension to the case $p = N$ of Proposition 4.1.

Proposition 4.3. *Under the assumptions of Theorem 3.10 there exists* $\lim_{k\to\infty} u_k = u_\infty$ *which is the unique element of* $C(\overline{\Omega} \setminus \{0\}) \cap C^1(\Omega)$ *which satisfies* (3.17) *in* Ω*, vanishes on* $\partial\Omega \setminus \{0\}$ *and such that*

$$\lim_{x\to 0} \frac{u_\infty(x)}{U_{s_+^{N-1}}(x)} = 1. \tag{4.93}$$

Proof. We denote by u_k^Ω the unique positive solution of (3.17) satisfying (3.57) obtained in Theorem 3.6. Then

$$T_\ell[u_k^\Omega] = u^{\Omega^\ell}_{\ell^{\beta_q - \beta_*}k}, \tag{4.94}$$

because of uniqueness. We denote by $B := B_{\frac{1}{2}}(a)$ and $B' := B_{\frac{1}{2}}(a')$ the two balls tangent to $\partial\Omega$ at 0 respectively interior and exterior to Ω introduced in the proof of Lemma 3.11. Estimate (3.58) implies

$$u_k^{B'^c} \leq u_k^\Omega \leq u_k^B \tag{4.95}$$

the left-hand side inequality holding in Ω and the right-hand side one in B. Therefore

$$T_\ell[u_k^{B'^c}] := u^{B'^c\,\ell}_{\ell^{\beta_q-\beta_*}k} \leq T_\ell[u_k^\Omega] \leq T_\ell[u_k^B] := u^{B^\ell}_{\ell^{\beta_q-\beta_*}k}, \tag{4.96}$$

the domains of validity of these inequalities being modified accordingly. Using again (3.58) we obtain

$$T_{\ell'}[u_{k'}^{B'^c}] \leq T_\ell[u_k^{B'^c}] \qquad \text{in } B'^{c\,\ell'}, \tag{4.97}$$

for any $0 < \ell' \leq \ell$ and $\ell'^{\beta_q-\beta_*}k' \leq \ell^{\beta_q-\beta_*}k$. In the same way

$$T_{\ell'}[u_{k'}^{B}] \geq T_\ell[u_k^{B}] \qquad \text{in } B^{\ell}, \tag{4.98}$$

for any $0 < \ell' \leq \ell$ and $\ell'^{\beta_q-\beta_*}k' \geq \ell^{\beta_q-\beta_*}k$. Since u_k^Ω u_k^B, $u_k^{B'^c}$ are increasing with respect to k, they converge respectively to u_∞^Ω u_∞^B, $u_\infty^{B'^c}$ and there holds for any $\ell > 0$

$$T_\ell[u_\infty^{B'^c}] \leq T_\ell[u_\infty^\Omega] \leq T_\ell[u_\infty^B], \tag{4.99}$$

from (4.96) and

$$\begin{array}{lll} (i) & T_{\ell'}[u_\infty^{B'^c}] \leq T_\ell[u_\infty^{B'^c}] & \text{in } B'^{c\,\ell'} \\ (ii) & T_{\ell'}[u_\infty^{B}] \geq T_\ell[u_\infty^{B}] & \text{in } B^{\ell} \end{array} \tag{4.100}$$

for any $0 < \ell' \leq \ell$. Notice that , replacing ℓ by $\ell\ell'$ we can rewrite (4.99) as follows

$$T_{\ell'}[T_\ell[u_\infty^{B'^c}]] \leq T_{\ell'}[T_\ell[u_\infty^\Omega]] \leq T_{\ell'}[T_\ell[u_\infty^B]]. \tag{4.101}$$

Because of the monotonicity with respect to ℓ the following limits exist

$$U^{B'^c} = \lim_{\ell\to 0} T_\ell[u_\infty^{B'^c}] \quad \text{and} \quad U^B = \lim_{\ell\to 0} T_\ell[u_\infty^B]. \tag{4.102}$$

By Lemma 2.5 applied with $\phi(|x|) = |x|^{-\beta_q}$ and since there holds $u_\infty^B(x) \leq c|x|^{-\beta_q}$ and $u_\infty^{B'}(x) \leq c|x|^{-\beta_q}$, we derive

$$\begin{array}{lll} (i) & |\nabla T_\ell[u_\infty^B](x)| \leq c_2|x|^{-\beta_q-1} & \forall x \in B^\ell \\ (ii) & |\nabla T_\ell[u_\infty^B](x) - \nabla T_\ell[u_\infty^B](y)| \leq c_2|x|^{-\beta_q-1-\alpha}|x-y|^\alpha & \forall x,y \in B^\ell,\ |x| \leq |y| \\ (iii) & T_\ell[u_\infty^B](x) \leq c_2|x|^{-\beta_q-1}(\text{dist}\,(x,\partial B^\ell))^\alpha & \forall x \in B^\ell, \end{array} \tag{4.103}$$

and

$$\begin{array}{lll} (i) & |\nabla T_\ell[u_\infty^{B'^c}](x)| \leq c_2|x|^{-\beta_q-1} & \forall x \in B'^{c\,\ell} \\ (ii) & |\nabla T_\ell[u_\infty^{B'^c}](x) - \nabla T_\ell[u_\infty^{B'^c}](y)| \leq c_2|x|^{-\beta_q-1-\alpha}|x-y|^\alpha & \forall x,y \in B'^{c\,\ell},\ |x| \leq |y| \\ (iii) & T_\ell[u_\infty^{B'^c}](x) \leq c_2|x|^{-\beta_q-1}(\text{dist}\,(x,\partial B'^{c\,\ell}))^\alpha & \forall x \in B'^{c\,\ell}. \end{array} \tag{4.104}$$

Thus the sets of functions $\{T_\ell[u_\infty^B]\}$ and $\{T_\ell[u_\infty^{B'}]\}$ are equicontinuous in the C^1-loc topology and by uniqueness, the limit in (4.102) below holds in this topology. Hence $U^{B'^c}$ and U^{B^c} are positive solutions of (3.17) in $\mathbb{R}^N_+$ which vanish on $\partial\mathbb{R}^N_+ \setminus \{0\}$. Furthermore $U^{B'^c} \leq U^{B^c}$ Since for any $\ell, \ell' > 0$,

$T_{\ell'}[T_\ell[u_\infty^{B'^c}]] = T_{\ell\ell'}[u_\infty^{B'^c}]$, it follows $T_{\ell'}[U^{B'^c}] = U^{B'^c}$ and in the same way $T_{\ell'}[U^B] = U^B$. This means that U^B and $U^{B'^c}$ are self-similar solutions of (3.17) in $\mathbb{R}_+^N$ and they vanish on $\partial\mathbb{R}_+^N \setminus \{0\}$. Hence

$$U^B = U^{B'^c} = U_{S_+^{N-1}}. \tag{4.105}$$

Applying again Lemma 2.5 to u_∞^Ω with $\phi(|x|) = |x|^{-\beta_q}$ we have

$$\begin{array}{lll} (i) & |\nabla T_\ell[u_\infty^\Omega](x)| \leq c_2|x|^{-\beta_q-1} & \forall x \in \Omega^\ell \\ (ii) & |\nabla T_\ell[u_\infty^\Omega](x) - \nabla T_\ell[u_k^\Omega](y)| \leq c_2|x|^{-\beta_q-1-\alpha}|x-y|^\alpha & \forall x, y \in \Omega^\ell,\ |x| \leq |y| \\ (iii) & T_\ell[u_\infty^\Omega](x) \leq c_2|x|^{-\beta_q-1}(\text{dist}\,(x, \partial\Omega^\ell))^\alpha & \forall x \in \Omega^\ell. \end{array} \tag{4.106}$$

This implies that the set of functions $\{T_\ell[u_\infty^\Omega]\}_\ell$ is equicontinuous in the C^1-loc topology of $\mathbb{R}_+^N$ and there exists a sequence $\{\ell_n\} \to 0$ and a function U such that $T_{\ell_n}[u_\infty^\Omega] \to U^\Omega$ in this topology of $\mathbb{R}_+^N$, and U is a positive solution of (3.17) in $\mathbb{R}_+^N$ which vanishes on $\partial\mathbb{R}_+^N \setminus \{0\}$. From (4.99) and (4.105) there holds $U^\Omega = U_{S_+^{N-1}}$ and therefore

$$\lim_{\ell\to 0} T_\ell[u_\infty^\Omega] = U_{S_+^{N-1}}. \tag{4.107}$$

This implies (4.93) and

$$\lim_{r\to 0} r^{\beta_q} u_\infty^\Omega(r,\sigma) = \omega_{S_+^{N-1}}(\sigma) \tag{4.108}$$

uniformly on compact subsets of S_+^{N-1}. $\square$

Up to minor modifications the proof of the next classification theorem is similar to the one of Theorem 4.2.

Theorem 4.4. *Assume $N-1 < q < N - \frac{1}{2}$ If $u \in C(\overline{\Omega} \setminus \{0\}) \cap C^1(\Omega)$ is a positive solution of (3.17) in Ω which vanishes on $\partial\Omega \setminus \{0\}$, then we have the following alternative:*

(i) either there exists $k \geq 0$ such that (4.74) holds,

(ii) or (4.75) holds.

5 Appendix I: Positive p-harmonic functions in a half space

In this section we prove the following rigidity result.

Theorem 5.1. *Assume $1 < p \leq N$ and $u \in C^1(\mathbb{R}_+^N) \cap C(\overline{\mathbb{R}_+^N} \setminus \{0\})$ is a positive p-harmonic function which vanishes on $\partial\mathbb{R}_+^N \setminus \{0\}$ and such that $|x|^{\beta_*} u(x)$ is bounded. Then there exists $k \geq 0$ such that*

$$u(x) = k\Psi_*(x) \qquad \forall x \in \mathbb{R}_+^N. \tag{5.1}$$

Proof. Since $|x|^{\beta_*} u(x)$ is bounded, $|x|^{\beta_*+1} \nabla u(x)$ is also bounded and there exists $m > 0$ such that $u(x) \leq m\Psi_*(x)$ in B_δ^+. We denote by k the infimum of the $c > 0$ such that $u(x) \leq c\Psi_*(x)$. Then

$$0 \leq u(x) \leq k\Psi_*(x) \qquad \forall x \in \mathbb{R}_+^N \setminus \{0\} \tag{5.2}$$

and we assume that $k > 0$ otherwise $u = 0$. Assume that the graphs over $\mathbb{R}^N_+$ of the functions $x \mapsto u(x)$ and $x \mapsto k\Psi_*(x)$ are tangent at some point $x_0 \in \mathbb{R}^N_+$ or $x_0 \in \partial\mathbb{R}^N_+ \setminus \{0\}$. Since $\nabla\Psi_*$ never vanishes in $\overline{\mathbb{R}}^N_+ \setminus \{0\}$ it follows from the strong maximum principle or Hopf Lemma that $u = k\Psi_*$. If the two graphs are not tangent in $\overline{\mathbb{R}}^N_+ \setminus \{0\}$, either they are asymptotically tangent at 0, or at ∞.

(i) In the first case there exists two sequences $\{k_n\}$ increasing to k and $\{x_n\} \subset \mathbb{R}^N_+$ converging to zero such that $\frac{u(x_n)}{\Psi_*(x_n)} = k_n$. We set $r_n = |x_n|$ and $u_{r_n}(x) = r_n^{\beta_*} u(r_n x)$. Then u_{r_n} is p-harmonic and positive and $0 < u_{r_n}(x) \leq k\,|x|^{-\beta_*}\,\psi_*(\frac{x}{|x|})$; therefore

$$|\nabla u_{r_n}(x)| \leq C\,|x|^{-\beta_*-1} \text{ and } \left|\nabla u_{r_n}(x) - \nabla u_{r_n}(x')\right| \leq C\,|x|^{-\beta_*-1-\alpha}\left|x - x'\right|^{\alpha} \tag{5.3}$$

for $0 < |x| \leq |x'|$ and some constants $C > 0$ and $\alpha \in (0,1)$. Up to a subsequence, we can assume that u_{r_n} converges to some U in the C^1_{loc} topology of $\overline{\mathbb{R}}^N_+ \setminus \{0\}$ and $\frac{x_n}{r_n} \to \xi \in S^{N-1}_+$. The function U is p-harmonic and positive in $\mathbb{R}^N_+$ and satisfies $0 \leq U \leq k\Psi_*$ in $\mathbb{R}^N_+$ and $U(\xi) = k\Psi_*(\xi)$ if $\xi \in S^{N-1}_+$ or $U_{x_N}(\xi) = k\Psi_{*\,x_N}(\xi)$ if $\xi \in \partial S^{N-1}_+$. It follows from the strong maximum principle or Hopf Lemma that $U = k\Psi_*$. Therefore $u_{r_n} \to k\Psi_*$ and in particular

$$\lim_{r_n \to 0} \frac{r_n^{\beta_*} u(r_n, \sigma)}{\psi_*(\sigma)} = k \quad \text{uniformly on } S^{N-1}_+. \tag{5.4}$$

For any $\epsilon > 0$, there exists $n_\epsilon \in \mathbb{N}_*$ such that for $n \geq n_\epsilon$, $(k - \epsilon)\Psi_*(x) \leq u(x) \leq (k + \epsilon)\Psi_*(x)$ if $|x| = r_n$. This implies $(k - \epsilon)\Psi_*(x) \leq u(x) \leq (k + \epsilon)\Psi_*$ for $|x| \geq r_n$ and therefore in $\mathbb{R}^N$. Since ϵ is arbitrary, we deduce that $u = k\Psi_*$.

(ii) if the two graphs are tangent at infinity, there exist two sequences $\{k_n\}$ increasing to k and $\{x_n\}$ such that $r_n = |x_n| \to \infty$ with $u(x_n) = k_n\Psi_*(x_n)$ and

$$\lim_{r_n \to \infty} \frac{r_n^{\beta_*} u(r_n, \sigma)}{\psi_*(\sigma)} = k \quad \text{uniformly on } S^{N-1}_+. \tag{5.5}$$

Therefore we look at the supremum of the $c > 0$ such that $u \geq c\Psi_*$. If the set of such c is empty, it would mean that

$$\inf_{x \in \mathbb{R}^N_+} \frac{u(x)}{\Psi_*(x)} = 0.$$

Clearly, if this infimum is achieved at some point, the strong maximum principle or Hopf Lemma imply $u \equiv 0$, contradicting (5.5), and this relation prevents also this infimum be achieved at infinity. We are left with the case where there exists a sequence $\{z_n\} \subset \mathbb{R}^N_+$, converging to 0, such that

$$\lim_{n \to \infty} \frac{u(z_n)}{\Psi_*(z_n)} = 0. \tag{5.6}$$

By boundary Harnack inequality [2, th 2.11], there exists $c > 0$ such that

$$c^{-1}\frac{u(z)}{\Psi_*(z)} \leq \frac{u(z_n)}{\Psi_*(z_n)} \leq c\frac{u(z)}{\Psi_*(z)} \quad \forall z \in \mathbb{R}^N_+ \text{ s.t. } |z| = |z_n| \tag{5.7}$$

Combining (5.6) and (5.7), we derive that

$$\lim_{n\to\infty}\sup_{|z|=|z_n|}\frac{u(z)}{\Psi_*(z)}=0, \tag{5.8}$$

Denoting by ϵ_n the supremum in the above relation, we obtain that $u\leq\epsilon_n\Psi_*$ in $\mathbb{R}^N_+\setminus B_{\epsilon_n}$ and finally $u=0$, contradiction. Thus we are left with the case where there exists $k'\in(0,k]$ which is the supremum of the $c>0$ such that $u\geq c\Psi_*$. In particular $u\geq k'\Psi_*$. Remembering that $u\leq k\Psi_*$ we get $k=k'$, which implies $u=k\Psi_*$.

Next we assume that $k'<k$. Clearly the graphs of u and $k'\Psi_*$ cannot be tangent in $\overline{\mathbb{R}}^N_+$, because of strong maximum principle or Hopf Lemma. They cannot be tangent at infinity because of (5.5). Therefore there exist two sequences $\{k'_n\}$ increasing to k' and $\{x'_n\}\subset\mathbb{R}^N_+$ converging to 0 such that $\frac{u(x'_n)}{\Psi_*(x'_n)}=k'_n$. As in case (i) we obtain that

$$\lim_{r'_n\to 0}\frac{r_n'^{\beta_*}u(r'_n,\sigma)}{\psi_*(\sigma)}=k'\quad\text{uniformly on }\;S^{N-1}_+, \tag{5.9}$$

where $r'_n=|x'_n|$, and finally derive that $u=k'\Psi_*$, a contradiction with (5.5). Therefore $k=k'$, which ends the proof. □

Remark. In the case $p=N$ the result holds under the weaker assumption $\lim\limits_{|x|\to\infty}u(x)=0$. This is due to the fact that this condition implies by regularity

$$\lim_{|x|\to\infty}\frac{u(x)}{\omega_{s^{N-1}_+}(\frac{x}{|x|})}=0$$

and therefore

$$u(x)\leq m\Psi_*(x)\quad\forall x\ \text{ s.t. }\ |x|\geq 1,$$

where $m=\max_{|x|=1}\dfrac{u(x)}{\omega_{s^{N-1}_+}(\frac{x}{|x|})}$. Using the inversion $x\mapsto\frac{x}{|x|^2}$, we obtain that the estimate $u\leq m\Psi_*$ holds $\mathbb{R}^N$, and we conclude by Theorem 5.1.

Remark. We conjecture that the rigidity result holds under the mere condition

$$\lim_{|x|\to\infty}|x|^{-\tilde\beta}u(x)=0, \tag{5.10}$$

were $\tilde\beta$ is the (positive) exponent corresponding to the regular spherical p-harmonic function under the form

$$\tilde\Psi=|x|^{\tilde\beta}\,\tilde\psi(\frac{x}{|x|}), \tag{5.11}$$

see [14], [12]. Note that $\tilde\beta=1$ when $p=N$.

6 Appendix II: Estimates on β_*

When $N=2$ and $1<p\leq 2$, it is proved in [9] that

$$\beta_*=\frac{3-p+2\sqrt{p^2-5p+7}}{3(p-1)}. \tag{6.1}$$

Up to now no estimate is known when $N>2$ except in the cases $p=2$ where $\beta_*=N-1$ and $p=N$ where $\beta_*=1$, besides the classical one

$$\beta_*>\frac{N-p}{p-1}, \tag{6.2}$$

valid when $p<N$. In this section we prove the following result

Theorem 6.1. *Assume $1<p<N$. Then the following estimates hold:*

$$1<p<2\Longrightarrow \beta_*>\frac{N-1}{p-1}, \tag{6.3}$$

$$2<p<N\Longrightarrow \max\left\{1,\frac{N-p}{p-1}\right\}<\beta_*<\frac{N-1}{p-1}. \tag{6.4}$$

Remark. It is worth noticing that when $p=2$ or $p=N$, there holds $\beta_*=\frac{N-1}{p-1}$.

Proof of Theorem 6.1. We consider the following set of spherical coordinates in $\mathbb{R}^N_+$ with $x=(x_1,...,x_N)$

$$\begin{aligned}
x_1 &= r\sin\theta_{N-1}\sin\theta_{N-2}...\sin\theta_2\sin\theta_1\\
x_2 &= r\sin\theta_{N-1}\sin\theta_{N-2}...\sin\theta_2\cos\theta_1\\
&\vdots\\
x_{N-1} &= r\sin\theta_{N-1}\cos\theta_{N-2}\\
x_N &= r\cos\theta_{N-1}
\end{aligned} \tag{6.5}$$

with $\theta_1\in[0,2\pi]$ and $\theta_k\in[0,\pi]$ for $k=2,...,N-2$ and $\theta_{N-1}\in[0,\frac{\pi}{2}]$. Under this representation, a solution ω of (3.2) verifies

$$\begin{aligned}
&-\frac{1}{\sin^{N-2}\theta_{N-1}}\left[\sin^{N-2}\theta_{N-1}\left(\beta_*^2\omega^2+\omega_{\theta_{N-1}}^2+\frac{1}{\sin^2\theta_{N-1}}\left|\nabla_{\theta'}\omega\right|^2\right)^{\frac{p-2}{2}}\omega_{\theta_{N-1}}\right]_{\theta_{N-1}}\\
&-\frac{1}{\sin^2\theta_{N-1}}div'_{\theta'}\left[\sin^{N-2}\theta_{N-1}\left(\beta_*^2\omega^2+\omega_{\theta_{N-1}}^2+\frac{1}{\sin^2\theta_{N-1}}\left|\nabla_{\theta'}\omega\right|^2\right)^{\frac{p-2}{2}}\nabla_{\theta'}\omega\right]\\
&=\beta_*\Lambda_{\beta_*}\left[\sin^{N-2}\theta_{N-1}\left(\beta_*^2\omega^2+\omega_{\theta_{N-1}}^2+\frac{1}{\sin^2\theta_{N-1}}\left|\nabla_{\theta'}\omega\right|^2\right)^{\frac{p-2}{2}}\omega\right]
\end{aligned} \tag{6.6}$$

where $\nabla_{\theta'}$ and $div'_{\theta'}$ denotes respectively the spherical gradient the divergence in variables $\theta'=(\theta_1,...,\theta_{N-2})$ parameterizing S^{N-2} and Λ_{β_*} is defined in Introduction. If ω is the unique positive solution of (3.2) (up to homothety), it depends only on θ_{N-1} and is C^∞. For simplicity we set $\theta_{N-1}=\theta\in[0,\frac{\pi}{2}]$ and $\omega=\omega(\theta)$ satisfies

$$\begin{array}{r}-\dfrac{1}{\sin^{N-2}\theta}\left[\sin^{N-2}\theta\left(\beta_*^2\omega^2+\omega_\theta^2\right)^{\frac{p-2}{2}}\omega_\theta\right]_\theta=\beta_*\Lambda_{\beta_*}\left[\sin^{N-2}\theta\left(\beta_*^2\omega^2+\omega_\theta^2\right)^{\frac{p-2}{2}}\omega\right]\\ \text{in } (0,\frac{\pi}{2})\\ \omega(\frac{\pi}{2})=0\,,\ \omega_\theta(0)=0.\end{array}\tag{6.7}$$

Step 1: The eigenvalue identity. Equation (6.7) can also be written under the form

$$-\omega_{\theta\theta}-(N-2)\cot\theta\,\omega_\theta-(p-2)\frac{\beta_*^2\omega+\omega_{\theta\theta}}{\beta_*^2\omega^2+\omega_\theta^2}\omega_\theta^2=\beta_*\Lambda_{\beta_*}\omega.\tag{6.8}$$

By multiplying (6.8) by $\cos\theta\sin^{N-2}\theta$ and then integrating over $(0,\frac{\pi}{2})$ we obtain

$$-\int_0^{\frac{\pi}{2}}\left(\omega_{\theta\theta}+(N-2)\cot\theta\,\omega_\theta\right)\cos\theta\sin^{N-2}\theta d\theta=(N-1)\int_0^{\frac{\pi}{2}}\omega\cos\theta\sin^{N-2}\theta d\theta.$$

Noticing that

$$\beta_*\Lambda_{\beta_*}+1-N=(p-1)\left(\beta_*-\frac{N-1}{p-1}\right)(\beta_*+1)$$

we derive

$$\begin{aligned}(2-p)&\int_0^{\frac{\pi}{2}}\frac{\beta_*^2\omega+\omega_{\theta\theta}}{\beta_*^2\omega^2+\omega_\theta^2}\omega_\theta^2\omega\cos\theta\sin^{N-2}\theta d\theta\\ &=(p-1)\left(\beta_*-\frac{N-1}{p-1}\right)(\beta_*+1)\int_0^{\frac{\pi}{2}}\omega\cos\theta\sin^{N-2}\theta d\theta.\end{aligned}\tag{6.9}$$

Step 2: Elliptic coordinates and reduction. Writing $\omega(\theta)=\omega(0)+a\theta^2+o(\theta^2)$, $\omega_\theta(\theta)=2a\theta+o(\theta)$ and $\omega_{\theta\theta}(\theta)=2a+o(1)$, then $-Na=\beta_*\Lambda_{\beta_*}$. This implies that ω is decreasing near 0. It is immediate that it cannot have a local minimum in $(0,\frac{\pi}{2})$, therefore it remains decreasing in the whole interval. We parameterize the ellipse

$$E_r=\{(x,y):x>0,\ y<0,\ x^2+\beta_*^{-2}y^2=r^2\}$$

by setting

$$\omega=r\cos\phi\ \text{ and }\ -\omega_\theta=\beta_*r\sin\phi\ \text{ with }\ \phi=\phi(\theta)\ \text{ and }\ r=r(\theta).$$

The functions r and ϕ are C^2. Hence $r_\theta\cos\phi-r\sin\phi\phi_\theta=-\beta_*r\sin\phi$, then $r_\theta\cos\phi=(\phi_\theta-\beta_*)r\sin\phi$ and $r_\theta=(\phi_\theta-\beta_*)r\tan\phi$. Plugging this into (6.8), we derive

$$-\left((p-1)\frac{r_\theta}{r}+\phi_\theta\cot\phi+(N-2)\cot\theta\right)+\Lambda_{\beta_*}\cot\phi=0,\tag{6.10}$$

and finally

$$(p-1)(\phi_\theta-\beta_*)\tan\phi+(\phi_\theta-\Lambda_{\beta_*})\cot\phi=(2-N)\cot\theta.\tag{6.11}$$

Step 3: Estimates on ϕ_θ. We can write (6.11) under the equivalent form

$$(p-1)(\phi_\theta-\beta_*)\tan^2\phi+\phi_\theta-\Lambda_{\beta_*}=(2-N)\frac{\cos\theta}{\cos\phi}\frac{\sin\phi}{\sin\theta}. \tag{6.12}$$

Since

$$\lim_{\theta\to 0}\frac{\sin\phi}{\sin\theta}=\lim_{\theta\to 0}\frac{\cos\phi}{\cos\theta}\phi_\theta=\phi_\theta(0),$$

we derive $\phi_\theta(0)-\Lambda_{\beta_*}=(2-N)\phi_\theta(0)$ and thus $\phi_\theta(0)=\dfrac{\Lambda_{\beta_*}}{N-1}$. Similarly, the expansion of $\phi(\theta)$ near $\theta=\frac{\pi}{2}$ yields to $\phi_\theta(\frac{\pi}{2})=\beta_*$. Since $p<N$, $\Lambda_{\beta_*}/(N-1)<\beta_*$. We claim now that

$$\phi_\theta(\theta)\leq\beta_* \qquad \forall\theta\in(0,\tfrac{\pi}{2}). \tag{6.13}$$

If $\Lambda_{\beta_*}\leq\beta_*$, then

$$(2-N)\cot\theta=(p-1)(\phi_\theta-\beta_*)\tan\phi+(\phi_\theta-\Lambda_{\beta_*})\cot\phi\geq((p-1)\tan\phi+\cot\phi)(\phi_\theta-\beta_*)$$

thus (6.13) holds.
Next we assume $\beta_*<\Lambda_{\beta_*}$. It means $0<(p-2)\beta_*-(N-p)$ and thus $p>2$. We claim that

$$\beta_*\leq\frac{N-2}{p-2}. \tag{6.14}$$

We proceed by contradiction and assume

$$\beta_*>\frac{N-2}{p-2}. \tag{6.15}$$

Then

$$(p-2)\left(\beta_*^2-\frac{N-p}{p-2}\beta_*-\frac{N-2}{p-2}\right)=(p-2)\left(\beta_*+1\right)\left(\beta_*-\frac{N-2}{p-2}\right)>0.$$

Equivalently

$$\beta_*(\Lambda_{\beta_*}-\beta_*)>N-2.$$

Since

$$\lim_{\theta\to\frac{\pi}{2}}\cot\theta\tan\phi=\lim_{\theta\to\frac{\pi}{2}}\frac{\cos\theta}{\cos\phi}=\lim_{\theta\to\frac{\pi}{2}}\frac{\sin\theta}{\phi_\theta\sin\phi}=\frac{1}{\beta_*}$$

and

$$\begin{aligned}(p-1)(\phi_\theta(\theta)-\beta_*)\tan^2\phi&=\Lambda_{\beta_*}-\phi_\theta(\theta)+(2-N)\frac{\cos\theta}{\cos\phi}\frac{\sin\phi}{\sin\theta}\\&=\frac{1}{\beta_*}\left(\beta_*(\Lambda_{\beta_*}-\beta_*)+2-N\right)+o(1),\end{aligned} \tag{6.16}$$

thus, if (6.15) holds there exists $\epsilon>0$ such that $\phi_\theta(\theta)>\beta_*$ for any $\theta\in[\frac{\pi}{2}-\epsilon,\frac{\pi}{2})$. Since $\phi_\theta(0)<\beta_*$, there exists $\bar\theta\in(0,\frac{\pi}{2})$ such that $\phi_\theta(\bar\theta)=\beta_*$ and $\phi_{\theta\theta}(\bar\theta)\geq 0$. We compute $\phi_{\theta\theta}$ and get

$$(p-1)\phi_\theta(\phi_\theta-\beta_*)\sec^2\phi+((p-1)\tan\phi+\cot\phi)\,\phi_{\theta\theta}-\phi_\theta(\phi_\theta-\Lambda_{\beta_*})\csc^2\phi=(N-2)\csc^2\theta$$

Hence, at $\theta = \bar\theta$

$$\phi_{\theta\theta}(\bar\theta)\left((p-1)\tan\phi(\bar\theta) + \cot\phi(\bar\theta)\right) = \beta_*(\beta_* - \Lambda_{\beta_*})\csc^2\phi(\theta) + (N-2)\csc^2\bar\theta$$

From (6.11),

$$\cot\phi(\bar\theta) = \frac{N-2}{\Lambda_{\beta_*} - \beta_*}\cot\bar\theta$$

Therefore

$$\begin{aligned}
A(\bar\theta) &:= \phi_{\theta\theta}(\bar\theta)\left((p-1)\tan\phi(\bar\theta) + \cot\phi(\bar\theta)\right)\\
&= \left(1 + \left(\frac{N-2}{\Lambda_{\beta_*} - \beta_*}\right)^2\cot^2\bar\theta\right)\beta_*(\beta_* - \Lambda_{\beta_*}) + (N-2)(1+\cot^2\bar\theta)\\
&= \beta_*(\beta_* - \Lambda_{\beta_*}) + N - 2 - \left(\frac{(N-2)^2}{\Lambda_{\beta_*} - \beta_*} + 2 - N\right)\cot^2\bar\theta\\
&= -(p-2)(\beta_*+1)\left(\beta_* - \frac{N-2}{p-2}\right) - \frac{N-2}{\Lambda_{\beta_*} - \beta_*}\left(\beta_*(N-1) - \Lambda_{\beta_*}\right)\cot^2\bar\theta\\
&< 0,
\end{aligned} \tag{6.17}$$

using (6.15) and the fact that $N > p$. This is a contradiction, thus (6.14) holds.

Next, if $\beta_* < \frac{N-2}{p-2}$, it follows from (6.16) that there exists $\epsilon > 0$ such that $\phi_\theta < \beta_*$ in $[\frac{\pi}{2} - \epsilon, \frac{\pi}{2})$. If (6.13) is not true, there exist $0 < \theta_1 < \theta_2 < \frac{\pi}{2} - \epsilon$ such that $\phi_\theta(\theta_1) = \phi_\theta(\theta_2) = \beta_*$, $\phi_{\theta\theta}(\theta_1) \geq 0$, $\phi_{\theta\theta}(\theta_2) \leq 0$. Using the equation satisfied by $\phi_{\theta\theta}$, we obtain for $i = 1, 2$,

$$A(\theta_i) = (2-p)(\beta_*+1)\left(\beta_* - \frac{N-2}{p-2}\right) - \frac{N-2}{\Lambda_{\beta_*} - \beta_*}(\beta_*(N-1) - \Lambda_{\beta_*})\cot^2\theta_i. \tag{6.18}$$

On one hand $A(\theta_2) \leq 0 \leq A(\theta_1)$, and on the other

$$A(\theta_2) - A(\theta_1) = \frac{N-2}{\Lambda_{\beta_*} - \beta_*}(\beta_*(N-1) - \Lambda_{\beta_*})(\cot^2\theta_1 - \cot^2\theta_2) > 0,$$

since cot is decreasing in $(0, \frac{\pi}{2})$, $\cot^2\theta_1 > \cot^2\theta_2$, a contradiction. Therefore $\phi_\theta \leq \beta_*$ in $(0, \frac{\pi}{2})$.

Finally, if $\beta_* = \frac{N-2}{p-2}$ and the maximum of ϕ_θ on $[0, \frac{\pi}{2})$ is larger than β_* and achieved at some $\bar\theta < \frac{\pi}{2}$ the exists $\theta_1 < \bar\theta$ such that $\phi_\theta(\theta_1) = \beta_*$ and $\phi_{\theta\theta}(\theta_1) \geq 0$. In that case

$$0 \leq A(\theta_1) = -\frac{N-2}{\Lambda_{\beta_*} - \beta_*}\left(\beta_*(N-1) - \Lambda_{\beta_*}\right)\cot^2\theta_1 < 0$$

which is again a contradictions.

Step 4: End of the proof. Since $r^2 = \beta_*^2\omega^2 + \omega_\theta^2$, $r_\theta = r(\phi_\theta - \beta_*)\tan\phi$, we have

$$rr_\theta = \left(\beta_*^2\omega + \omega_{\theta\theta}\right)\omega_\theta = r(\phi_\theta - \beta_*)\tan\phi.$$

Since $\omega_\theta < 0$ on $(0, \frac{\pi}{2})$, it follows from Step 3 that $\beta_*^2\omega + \omega_{\theta\theta} \geq 0$ and thus

$$\int_0^{\frac{\pi}{2}} \frac{\beta_*^2\omega + \omega_{\theta\theta}}{\beta_*^2\omega^2 + \omega_\theta^2}\omega_\theta^2\omega\cos\theta\sin^{N-2}\theta d\theta > 0,$$

since the integrand cannot be identically 0. The conclusion follows from (6.9). □

Remark. Since $\omega_\theta(\frac{\pi}{2}) = -c^2 < 0$, it follows $\omega(\theta) = -\omega_\theta(\theta)\cot\theta + O(\frac{\pi}{2} - \theta)$ as $\theta \to \frac{\pi}{2}$, and from the eigenfunction equation (6.8)

$$\frac{\beta_*^2\omega + \omega_{\theta\theta}}{\beta_*^2\omega^2 + \omega_\theta^2}\omega_\theta^2 = (\beta_*^2\omega + \omega_{\theta\theta})(1 + o(1)).$$

Therefore

$$-(p-1)\omega_{\theta\theta} = (\beta_*\Lambda_{\beta_*} + (p-2)\beta_*^2 + 2 - N)\omega(1 + o(1)) \quad \text{as } \theta \to \frac{\pi}{2}$$

and since $\Delta'\omega := \omega_{\theta\theta} + (N-2)\cot\theta\,\omega_\theta$

$$-\Delta'\omega = \frac{\beta_*(\beta_*(2p-3) + p - N) + (p-2)(N-2)}{p-1}\omega(1 + o(1)) \quad \text{as } \theta \to \frac{\pi}{2}.$$

Because ω is C^∞ we obtain finally

$$\left|\Delta'\omega\right| \leq c\omega, \tag{6.19}$$

for some $c > 0$.

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